Dushnik-Miller Dimension and Boolean Dimension

William T. Trotter

trotter@math.gatech.edu

May 27, 2022

Georgia Tech Mathematics
College of Sciences

Realizers and Dushnik-Miller Dimension

- Let $t \geq 1$, and let $(L_1, \ldots, L_t)$ of linear orders on the ground set of P. For a distinct pair (x, y) of elements of P, define a query string $q(x, y)$ where bit i is 1 if $x < y$ in L_i; and 0 if $x > y$ in L_i.

Realizers and Dushnik-Miller Dimension

- Let $t \geq 1$, and let $(L_1, \ldots, L_t)$ of linear orders on the ground set of P. For a distinct pair (x, y) of elements of P, define a query string $q(x, y)$ where bit i is 1 if $x < y$ in L_i; and 0 if $x > y$ in L_i.
- Then $(L_1, \ldots, L_t)$ is a realizer (in the Dushnik-Miller) sense when the query string for (x, y) is $(1, 1, \ldots, 1)$ if and only if $x <_P y$.

Realizers and Dushnik-Miller Dimension

- Let $t \geq 1$, and let $(L_1, \ldots, L_t)$ of linear orders on the ground set of P. For a distinct pair (x, y) of elements of P, define a query string $q(x, y)$ where bit i is 1 if $x < y$ in L_i; and 0 if $x > y$ in L_i.
- Then $(L_1, \ldots, L_t)$ is a realizer (in the Dushnik-Miller) sense when the query string for (x, y) is $(1, 1, \ldots, 1)$ if and only if $x <_P y$.
- The Dushnik-Miller dimension of a poset P, denoted $\dim(P)$, is the least t such that P has a realizer of length t.

Examples of Deep and Beautiful Results

- Schnyder 1989 A graph G is planar if and only if the dimension of the poset of its vertices and edges ordered by inclusion has dimension at most 3.

Examples of Deep and Beautiful Results

- Schnyder 1989 A graph G is planar if and only if the dimension of the poset of its vertices and edges ordered by inclusion has dimension at most 3.
- Scott and Wood 2018 Let P be a poset in which each point is comparable with at most k other points. Then

$$\dim(P) \leq (2e^3 + o(1))(k \log k)e^{2\sqrt{\log \log k}}.$$

Examples of Deep and Beautiful Results

- Schnyder 1989 A graph G is planar if and only if the dimension of the poset of its vertices and edges ordered by inclusion has dimension at most 3.
- Scott and Wood 2018 Let P be a poset in which each point is comparable with at most k other points. Then

$$\dim(P) \leq (2e^3 + o(1))(k \log k)e^{2\sqrt{\log \log k}}.$$

- Erdős, Kierstead and WTT 1992 There exists a constant $c > 0$, such that the expected value of the dimension of a bipartite (n, n) poset is at least $n - cn/\log n$.

Dimension and Standard Examples

- Remark A large standard example forces large dimension, but large dimension does not require a large standard example.

Dimension and Standard Examples

- Remark A large standard example forces large dimension, but large dimension does not require a large standard example.
- Biró, Hamburger, Pór, and WTT 2016 There is a function $f : N \to N$ such that if $c \geq 1$, $|P| = 2n + 1$, and $\dim(P) \geq n - c$, then P contains a standard example S_d with $d \geq n - f(c)$.

Dimension and Standard Examples

- Remark A large standard example forces large dimension, but large dimension does not require a large standard example.
- Biró, Hamburger, Pór, and WTT 2016 There is a function $f : N \to N$ such that if $c \geq 1$, $|P| = 2n + 1$, and $\dim(P) \geq n - c$, then P contains a standard example S_d with $d \geq n - f(c)$.
- Conjecture Among posets with planar cover graphs, large dimension requires a large standard example.

Practical Motivation for Dimension ???

- Remark Perhaps a realizer (if it's small) provides an efficient way to answer a question of the form: Given a distinct pair (x, y) of elements of a poset P, is it true that $x <_P y$?

Practical Motivation for Dimension ???

- Remark Perhaps a realizer (if it's small) provides an efficient way to answer a question of the form: Given a distinct pair (x, y) of elements of a poset P, is it true that $x <_P y$?
- Remark If that's the end goal, then why not use a more flexible notion of a realizer?

Practical Motivation for Dimension ???

- Remark Perhaps a realizer (if it's small) provides an efficient way to answer a question of the form: Given a distinct pair (x, y) of elements of a poset P, is it true that $x <_P y$?
- Remark If that's the end goal, then why not use a more flexible notion of a realizer?
- Basic Idea Wouldn't it be enough to say that we can tell whether $x <_P y$ or not based entirely on the query string, i.e., some query strings are associated with a positive answer while others are associated with a negative answer.

An Informal Introduction to Boolean Dimension

- Nešetřil and Pudlák 1986 The Boolean dimension of a poset P, denoted $\mathrm{bdim}(P)$, is the least t such that there is a sequence $(L_1, \ldots, L_d)$ of linear orders on the ground set and a partition of the bit strings of length d as YES $\cup$ NO, so that for every pair (x, y) of distinct elements of P, $x <_P y$ if and only if $q(x, y)$ is in YES.

An Informal Introduction to Boolean Dimension

- Nešetřil and Pudlák 1986 The Boolean dimension of a poset P, denoted $\mathrm{bdim}(P)$, is the least t such that there is a sequence $(L_1, \ldots, L_d)$ of linear orders on the ground set and a partition of the bit strings of length d as YES $\cup$ NO, so that for every pair (x, y) of distinct elements of P, $x <_P y$ if and only if $q(x, y)$ is in YES.
- $\mathrm{bdim}(P) \leq \dim(P)$ for every poset P.

Working with Boolean Dimension I

- The Boolean dimension of a standard example is at most 4.

Working with Boolean Dimension I

- The Boolean dimension of a standard example is at most 4.
- Exercise $\mathrm{bdim}(S_d) = d$ when $d = 2, 3$, and $\mathrm{bdim}(S_d) = 4$, when $d \geq 4$.

Working with Boolean Dimension I

- The Boolean dimension of a standard example is at most 4.
- Exercise $\text{bdim}(S_d) = d$ when $d = 2, 3$, and $\text{bdim}(S_d) = 4$, when $d \geq 4$.
- Among posets with n points, there are some that have Boolean dimension at least $\Omega(\log n)$.

Working with Boolean Dimension I

- The Boolean dimension of a standard example is at most 4.
- Exercise $\mathrm{bdim}(S_d) = d$ when $d = 2, 3$, and $\mathrm{bdim}(S_d) = 4$, when $d \geq 4$.
- Among posets with n points, there are some that have Boolean dimension at least $\Omega(\log n)$.
- The Boolean dimension of the subset lattice B_n is $\Omega(n/\log n)$.

Working with Boolean Dimension I

- The Boolean dimension of a standard example is at most 4.
- Exercise $\mathrm{bdim}(S_d) = d$ when $d = 2, 3$, and $\mathrm{bdim}(S_d) = 4$, when $d \geq 4$.
- Among posets with n points, there are some that have Boolean dimension at least $\Omega(\log n)$.
- The Boolean dimension of the subset lattice B_n is $\Omega(n/\log n)$.
- Exercise When $2 \leq k < n$, let $P(1, k; n)$ be the poset of 1-element and k-element subsets of $[n]$ ordered by inclusion. Then $\mathrm{bdim}(1, k; n) \leq 2k$, and this is tight when n is large.

Working with Boolean Dimension II

- Among the interval orders, there are posets with large Boolean dimension

Working with Boolean Dimension II

- Among the interval orders, there are posets with large Boolean dimension
- Exercise There is a constant C such that if P is an interval order and Q is the split of P, then $\mathrm{bdim}(Q) \leq C$.

Working with Boolean Dimension II

- Among the interval orders, there are posets with large Boolean dimension
- Exercise There is a constant C such that if P is an interval order and Q is the split of P, then $\mathrm{bdim}(Q) \leq C$.
- There are interval orders that have large Boolean dimension

Working with Boolean Dimension II

- Among the interval orders, there are posets with large Boolean dimension
- Exercise There is a constant C such that if P is an interval order and Q is the split of P, then $\mathrm{bdim}(Q) \leq C$.
- There are interval orders that have large Boolean dimension
- Exercise For every $w \geq 1$, if P is a poset and the width of $P - \mathrm{Max}(P)$ is w, then $\mathrm{bdim}(P) \leq \dim(P) \leq w + 1$. Show that there is such a poset with $\mathrm{bdim}(P) = w + 1$.

Second Major Conjecture

- Nešetřil and Pudlák 1989 There is a constant C such that if P is a poset with a planar cover graph, then $\mathrm{bdim}(P) \leq C$.

Second Major Conjecture

- Nešetřil and Pudlák 1989 There is a constant C such that if P is a poset with a planar cover graph, then $\mathrm{bdim}(P) \leq C$.
- Micek, Smith-Blake, and WTT 2022+ Let P be a poset with a planar cover graph and a unique minimal element.

Second Major Conjecture

- Nešetřil and Pudlák 1989 There is a constant C such that if P is a poset with a planar cover graph, then $\operatorname{bdim}(P) \leq C$.
- Micek, Smith-Blake, and WTT 2022+ Let P be a poset with a planar cover graph and a unique minimal element. Then

$$\dim(P) \leq 2\,\operatorname{se}(P) + 2.$$

Second Major Conjecture

- Nešetřil and Pudlák 1989 There is a constant C such that if P is a poset with a planar cover graph, then $\mathrm{bdim}(P) \leq C$.
- Micek, Smith-Blake, and WTT 2022+ Let P be a poset with a planar cover graph and a unique minimal element. Then

$$\dim(P) \leq 2\,\mathrm{se}(P) + 2.$$

Also,

$$\mathrm{bdim}(P) \leq 13.$$