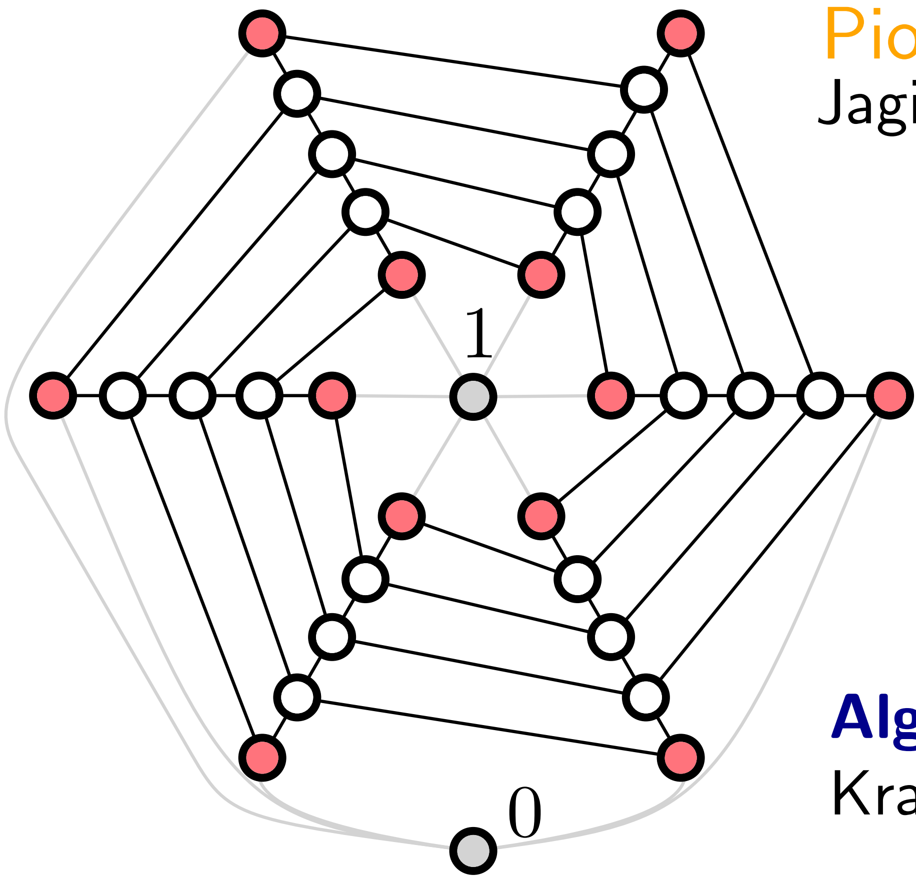
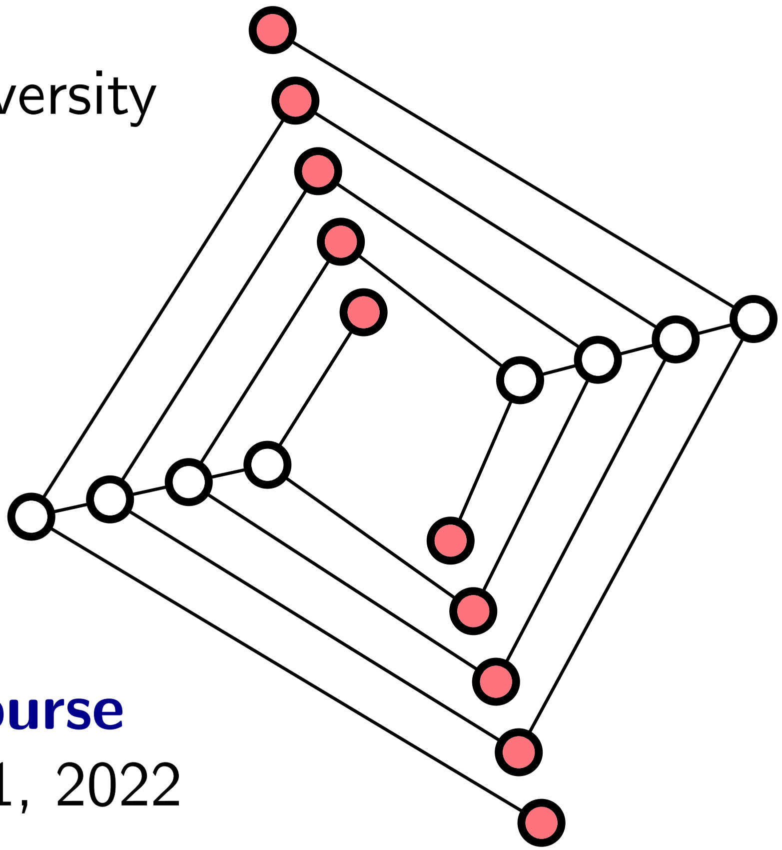
Combinatorics of posets



Piotr Micek
Jagiellonian University



AlgoMaNet course

Kraków, May 21, 2022

$P = (X, \leqslant)$	partial order / poset / order
X	ground set
$\leqslant$	binary relation on X $\left\{\begin{array}{l}\text{reflexive}\\ \text{antisymmetric}\\ \text{transitive}\end{array}\right.$

$P = (X, \leqslant)$ partial order / poset / order

X ground set

$\leqslant$ binary relation on X $\left\{ \begin{array}{l} \text{reflexive} \\ \text{antisymmetric} \\ \text{transitive} \end{array} \right.$

Example $X = \{a, b, c, d, e, f\}$

$$\leqslant = \{ab, ae, af, cd, ce, cf, de, df, ef\}$$

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X ground set

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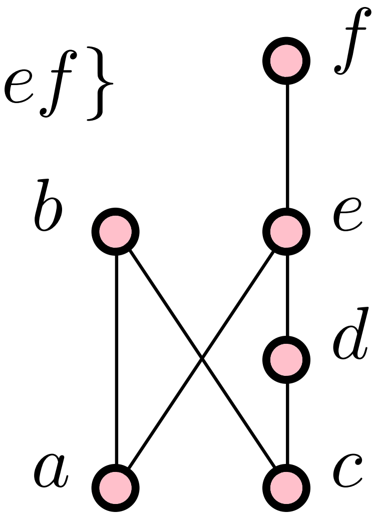

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X ground set

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x, y elements of P

(x, y) comparable if $x \leqslant y$ or $y \leqslant x$ in P

incomparable otw

cover relation if $x < y$ in P and there is no z st $x < z < y$ in P

$P = (X, \leqslant)$ partial order / poset / order

X ground set

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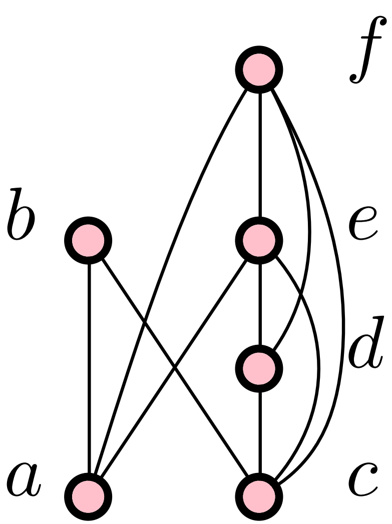
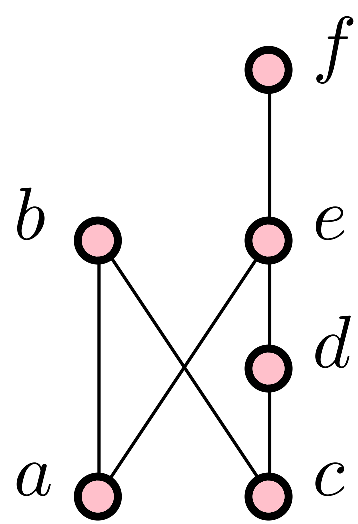
comparability graph



cover graph



diagram

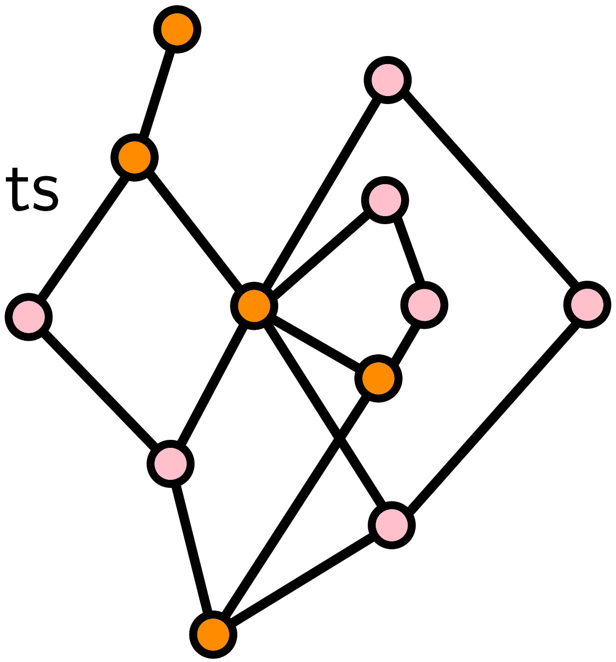
is a drawing of the cover graph such that

$\forall_{(x,y)}$ cover in P

x is drawn below y and xy-edge is vertically-monotone

chain a set of pairwise comparable elements

height the largest size of a chain

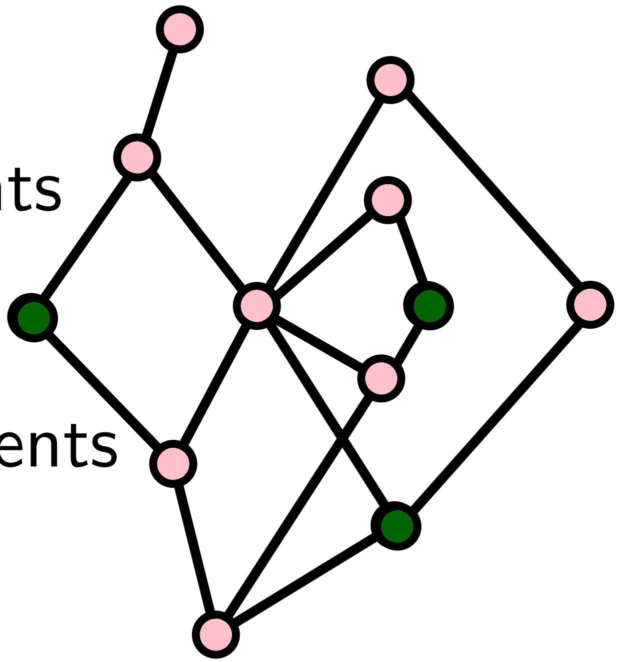


chain a set of pairwise comparable elements

height the largest size of a chain

antichain a set of pairwise incomparable elements

width the largest size of an antichain



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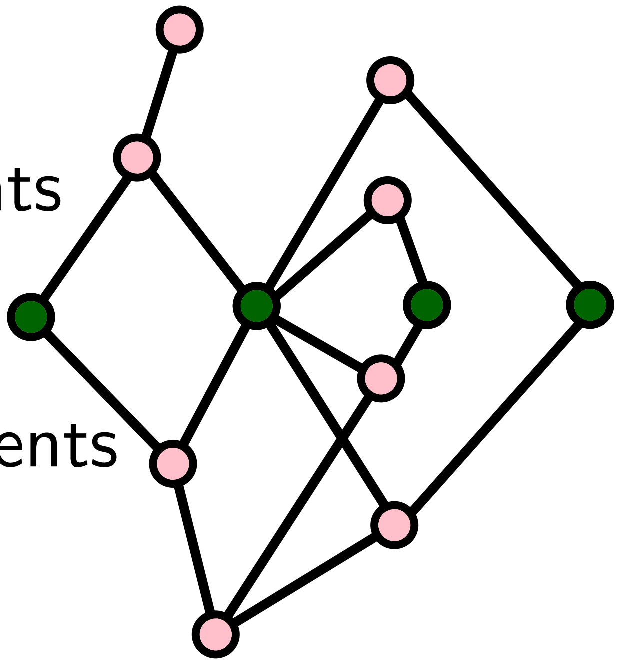

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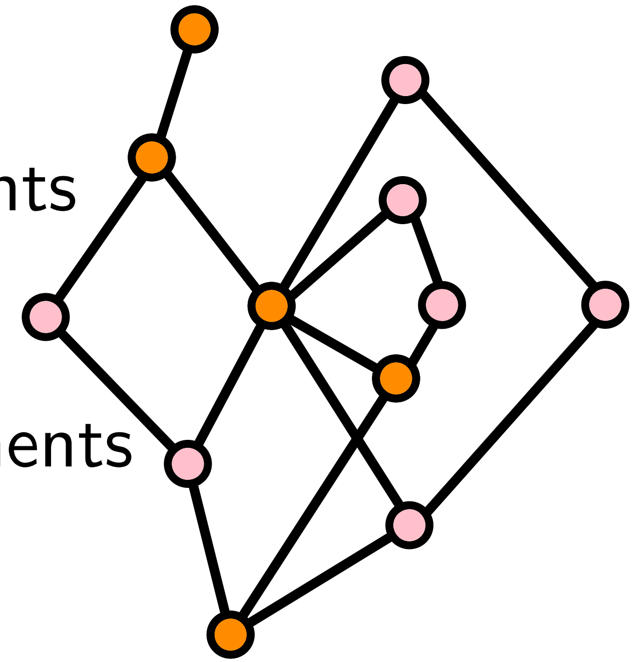
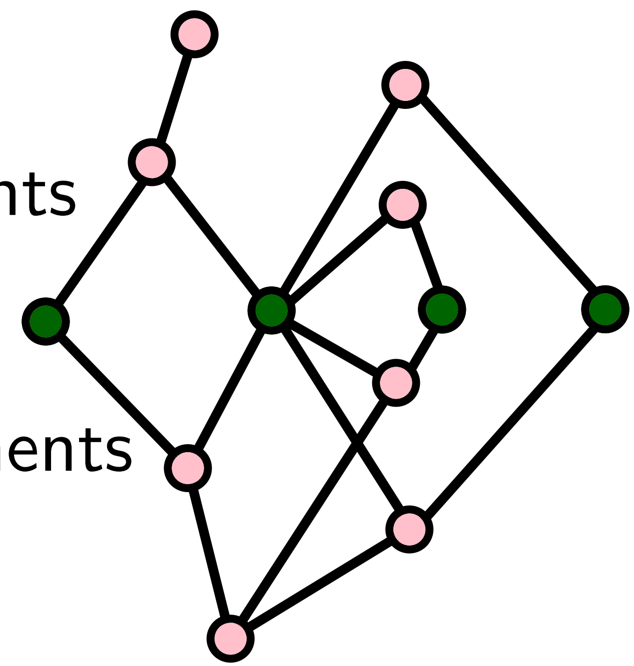


Observation Let P be a poset of height h.
Elements of P can be partitioned into h antichains



chain — a set of pairwise comparable elements

height — the largest size of a chain

antichain — a set of pairwise incomparable elements

width — the largest size of an antichain

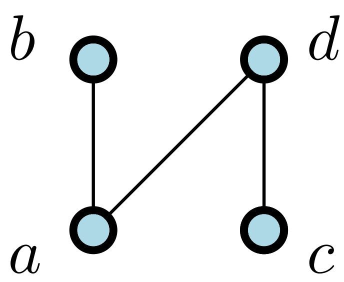
Observation Let P be a poset of height h.
Elements of P can be partitioned into h antichains

Theorem (Dilworth 1950) Let P be a poset of width w.
Elements of P can be partitioned into w chains

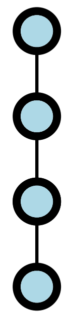
A linear extension L of a poset P is a linear order on the same ground set as P such that

$$x \leqslant y \text{ in } P \quad \Longrightarrow \quad x \leqslant y \text{ in } L \qquad \forall_{\,x,\, y \text{ in } P}$$

Example



P



d	d	b	d	b
c	b	d	b	d
b	c	c	a	a
a	a	a	c	c

all 5 linear extensions of P

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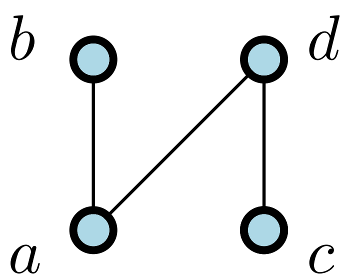

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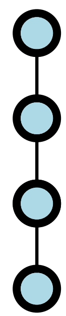

d	d	b	d	b
c	b	d	b	d
b	c	c	a	a
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all 5 linear extensions of P

Fact Every (finite) poset has a linear extension

Generic Algorithm

Input: P poset on n elements

for $i := 1$ to n do
  choose x_i a minimal element of $P - \{x_1, \ldots, x_{i-1}\}$
return $(x_1, \ldots, x_n)$

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Fact Every finite poset P has a minimal element.
Moreover, for every x in P there is $y \leqslant x$ in P such that y is minimal in P

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1. the algorithm is well-defined
2. it produces a linear extension
3. every linear extension of P is a possible outcome

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($\star$) room for specializations

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Corollary $$P = \bigcap_{\substack{L \text{ linear extension} \\ \text{of } P}} L$$

The dimension of a poset P, denoted $\dim(P)$, is the least positive integer d such that there are d linear extensions $L_1, \ldots, L_d$ of P such that

$$P = \bigcap_{i \in [d]} L_i$$

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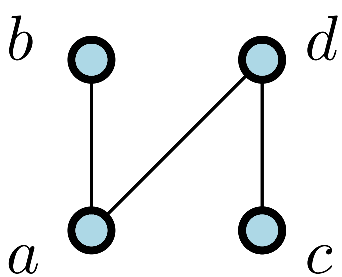
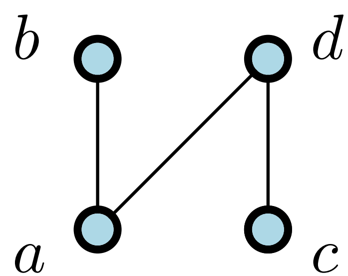
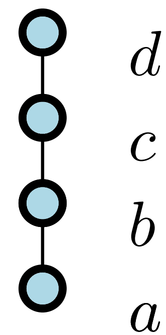
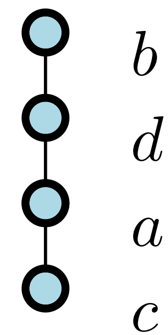
Example

$\dim\Bigg($ 

$\Bigg) \leqslant 2$ as 

$=$ 

$\cap$ 

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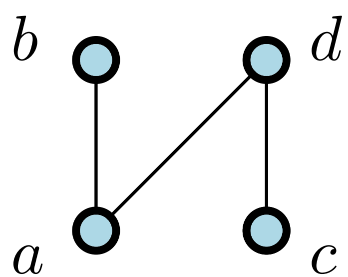
$\dim\Big($ 

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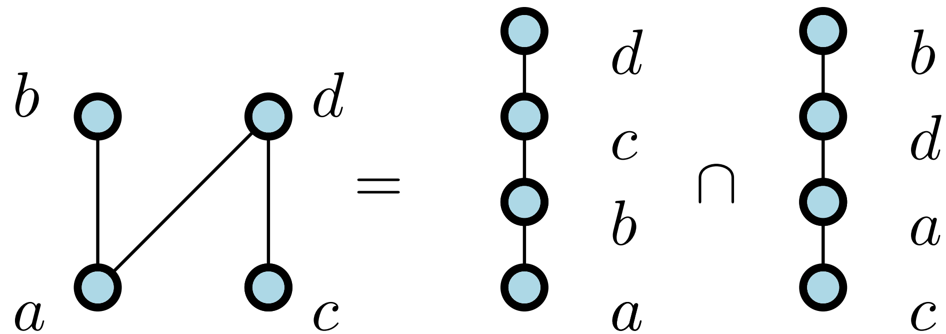
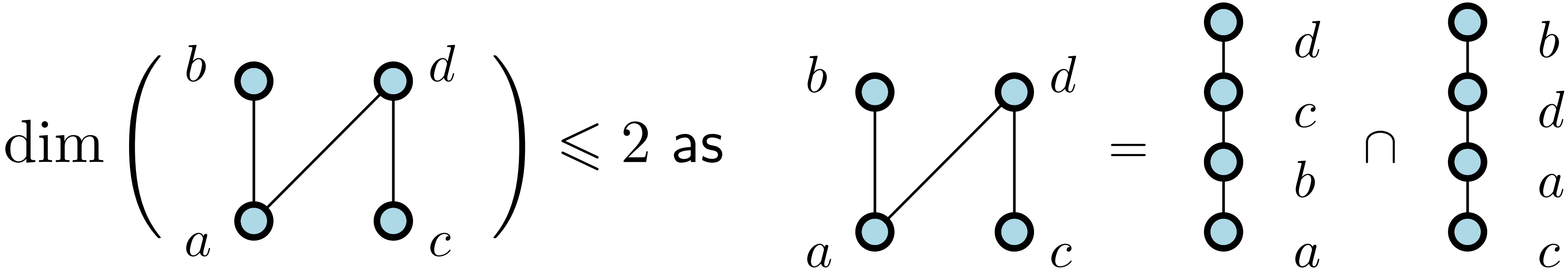
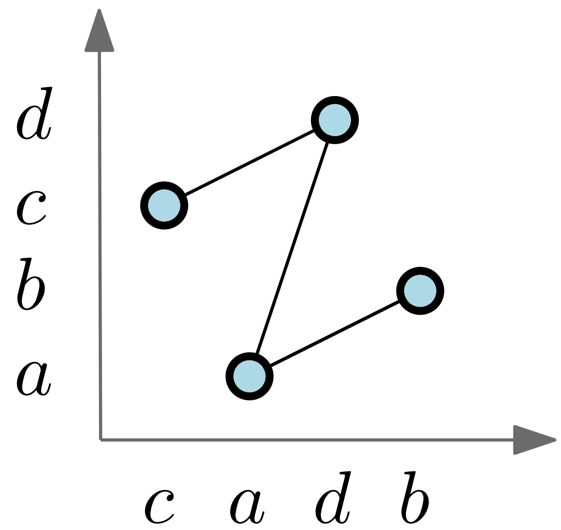


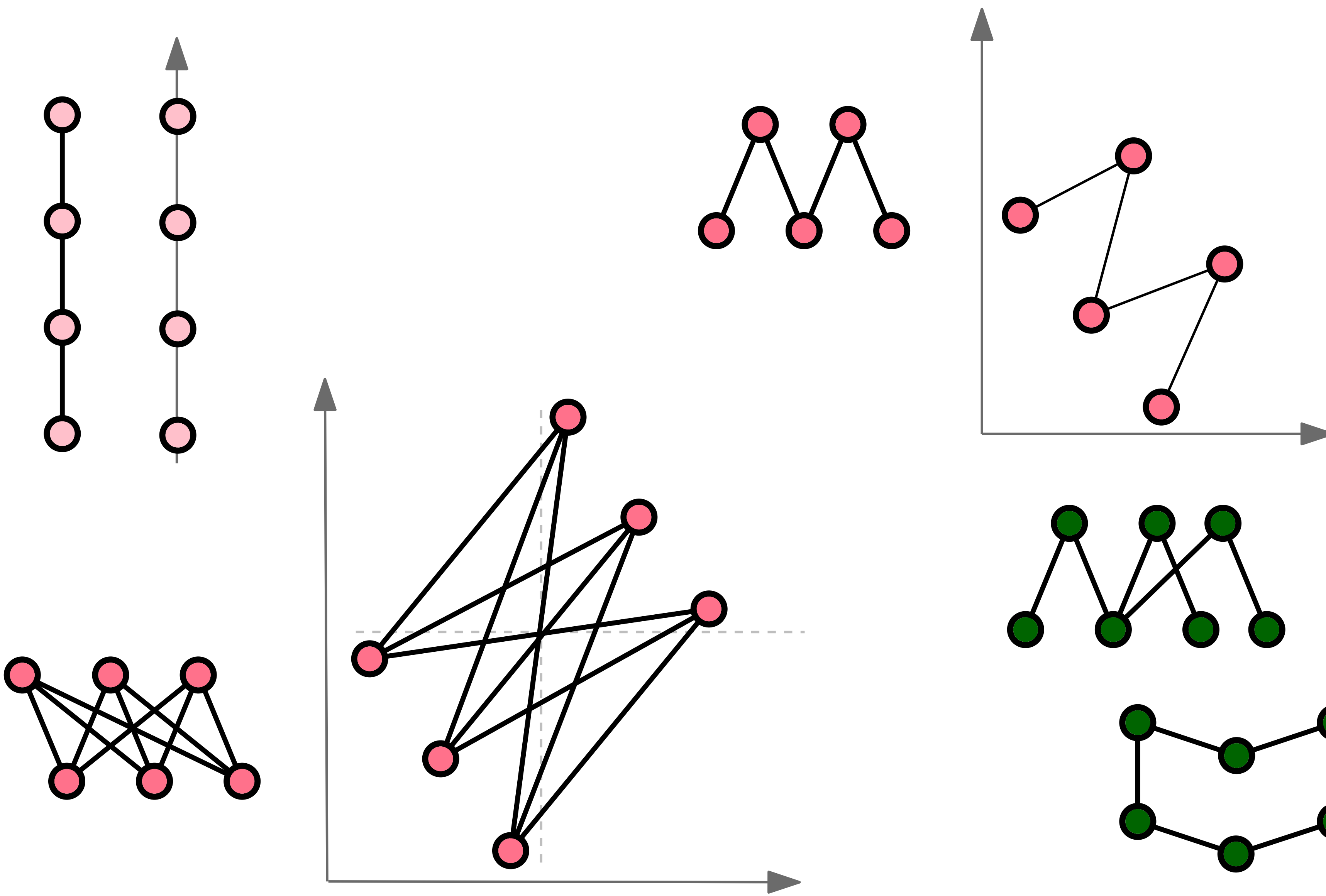
A set R of linear extensions of P such that $P = \bigcap_{L \in R} L$ is a realizer of P

Equivalently, dimension of a poset P is the least integer d such that elements of P can be embedded into $\mathbb{R}^d$ so that

$$x \leqslant y \text{ in } P \quad \iff \quad \text{point}(x) \text{ is below point}(y) \text{ in } \mathbb{R}^d$$







Exercise The following statements are equivalent:

(i) G is a comparability graph of a poset of dimension $\leqslant 2$

(ii) G is a containment graph of intervals on a line

(iii) G is a permutation graph

(iv) G and $\overline{G}$ are both comparability graphs

Exercise The following statements are equivalent:

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given a poset P

recognizing if $\dim(P) \leqslant 2$ can be done in polynomial time

recognizing if $\dim(P) \leqslant 3$ is NP-COMPLETE

Question Are there posets of large dimension?

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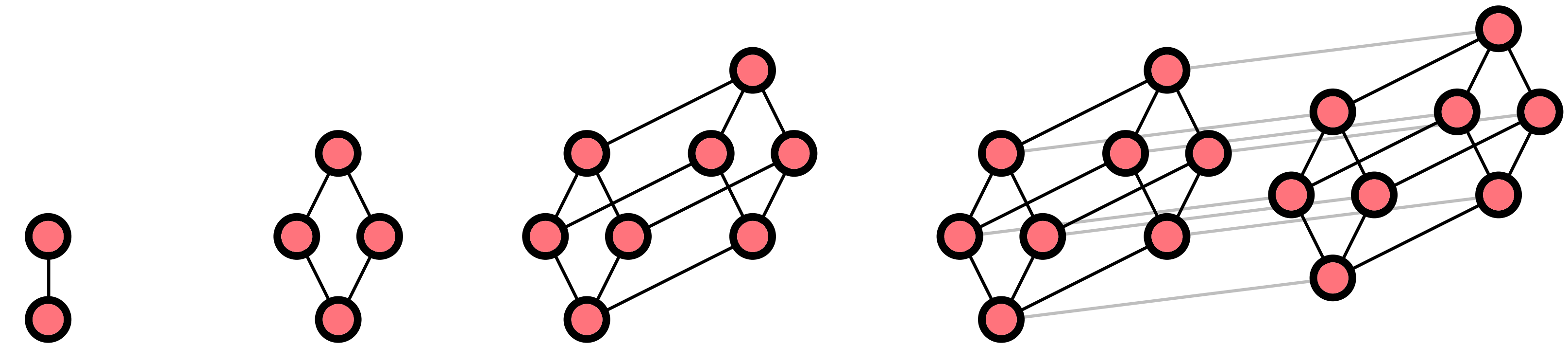
Example Dimension of Boolean lattices

For every $n \geqslant 1$, $\dim(\mathcal{B}_n) = n$



$\mathcal{B}_1$ $\mathcal{B}_2$ $\mathcal{B}_3$ $\mathcal{B}_4$

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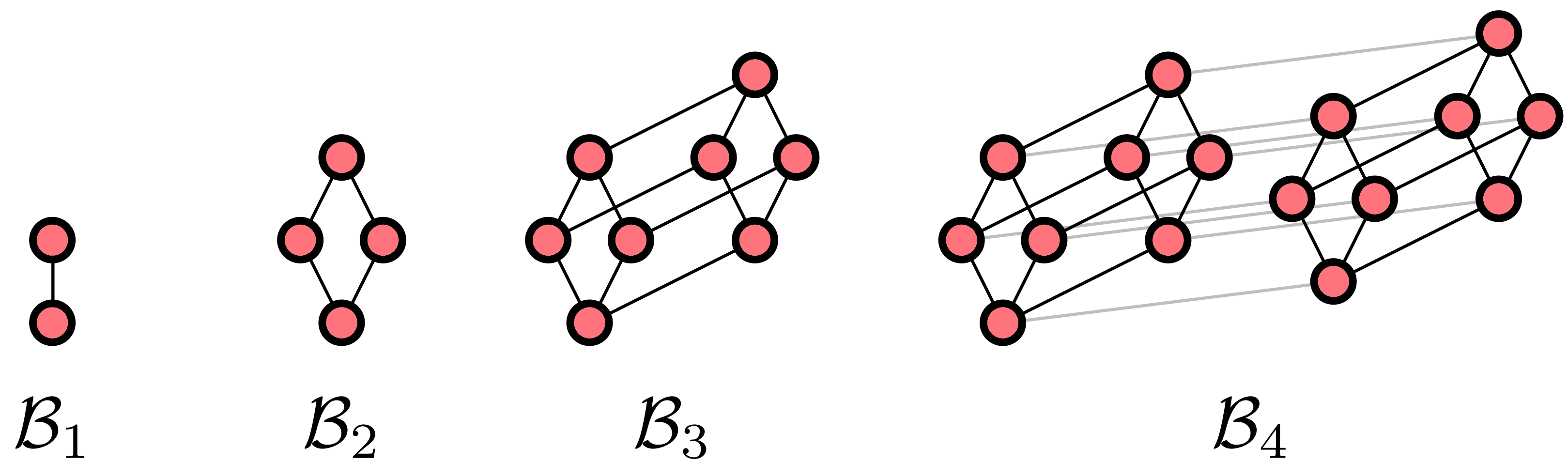

upper bound $L_1, \ldots, L_n$ linear extensions of $\mathcal{B}_n$

L_i produced by the generic algorithm
($\star$) prefer elements X with $i \notin X$

lower bound $\dim(\mathcal{B}_n) \geqslant n$

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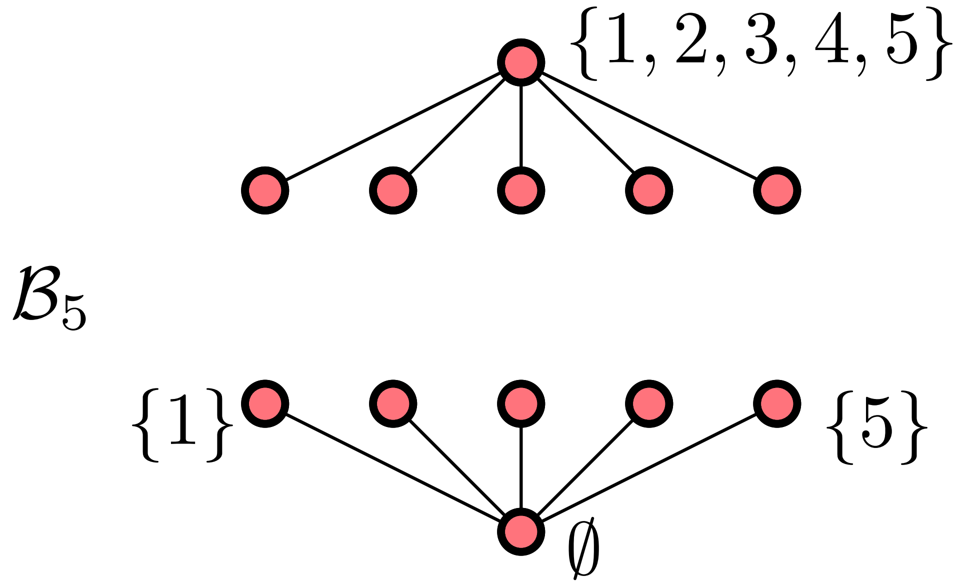
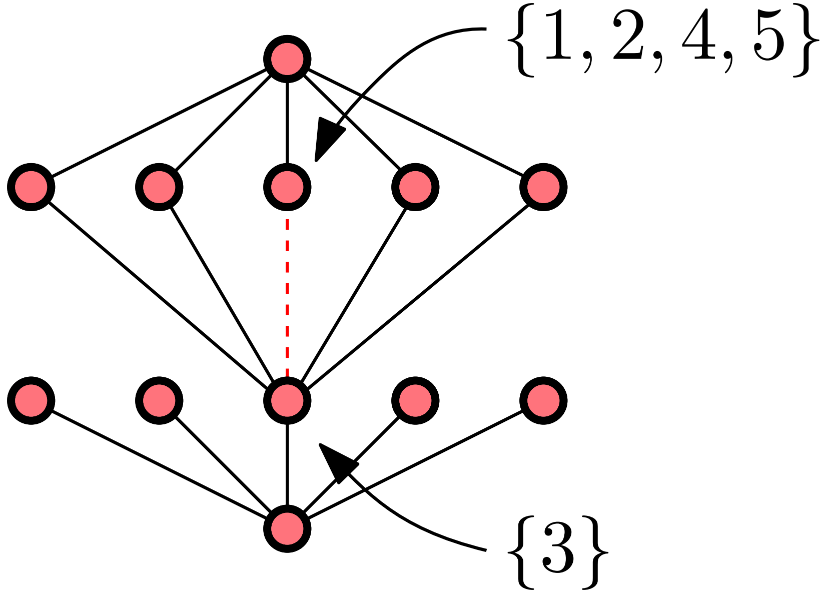


lower bound $\dim(\mathcal{B}_n) \geqslant n$

$\mathcal{B}_5$



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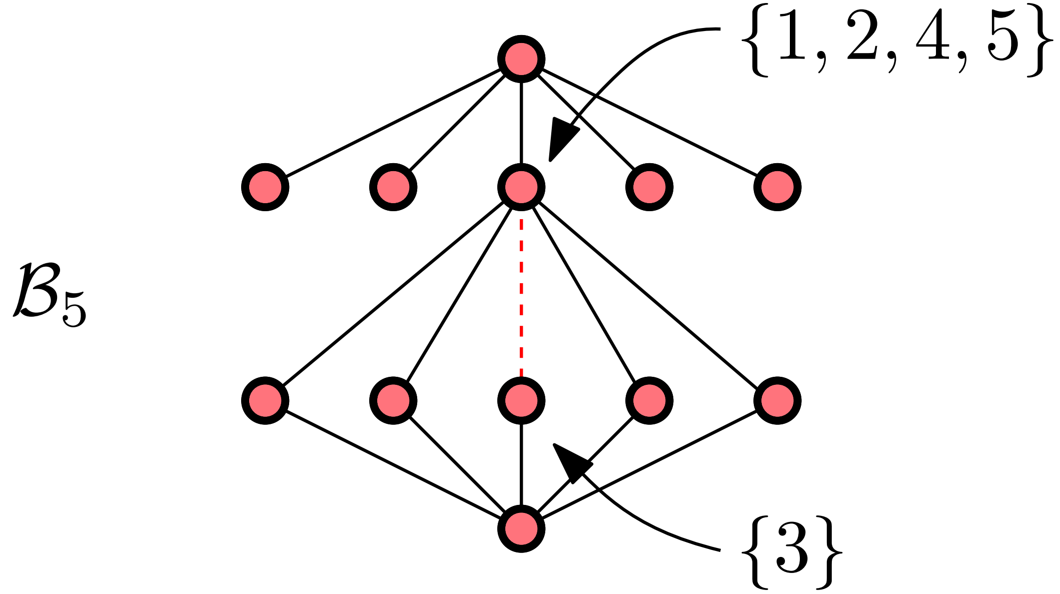
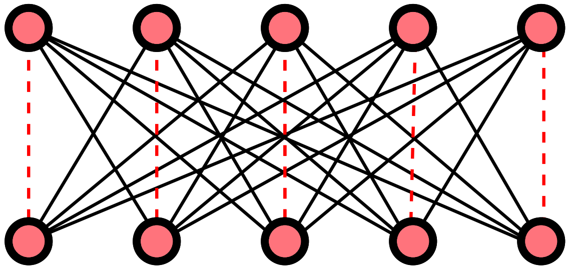


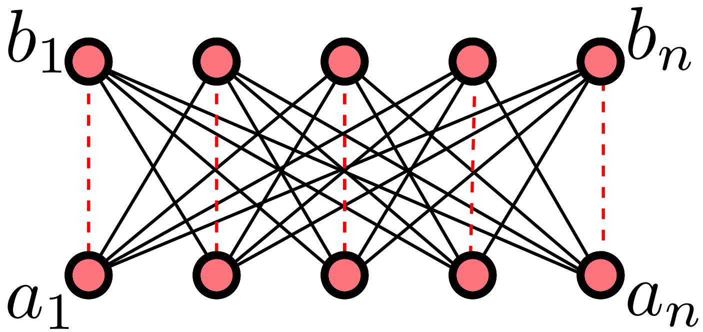
lower bound $\dim(\mathcal{B}_n) \geqslant n$

S_5 standard example of order 5



lower bound $\dim(\mathcal{B}_n) \geqslant n$

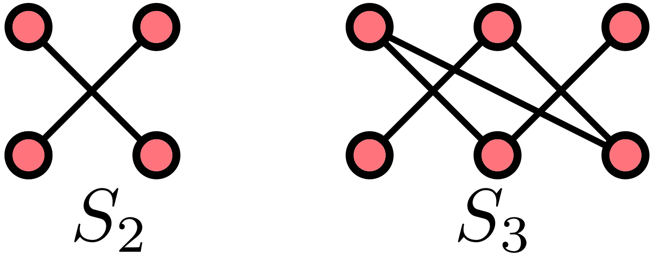
S_n standard example of order n



$2n$ elements

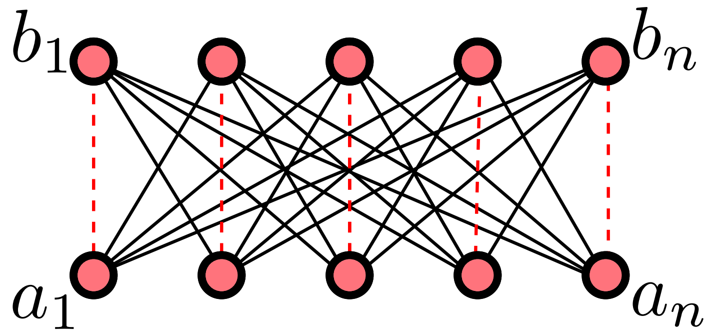
$\{a_1, \ldots, a_n\}$ and $\{b_1, \ldots, b_n\}$ are antichains

$a_i < b_j$ in $S_n \iff i \neq j$



lower bound $\dim(\mathcal{B}_n) \geqslant n$

S_n standard example of order n



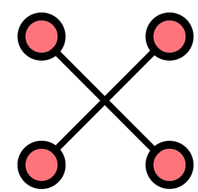
$2n$ elements

$\{a_1, \ldots, a_n\}$ and $\{b_1, \ldots, b_n\}$ are antichains

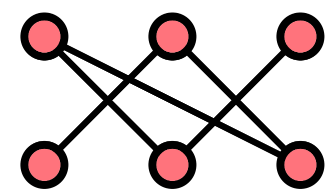
$a_i < b_j$ in $S_n \iff i \neq j$

Fact For every $n \geqslant 2$, $\dim(S_n) = n$

Proof (blackboard)



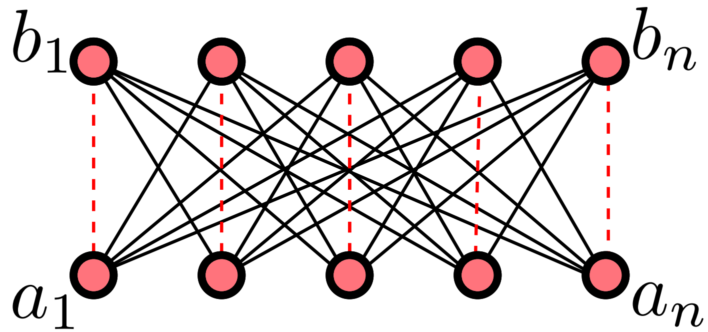
S_2



S_3

lower bound $\dim(\mathcal{B}_n) \geqslant n$

S_n standard example of order n



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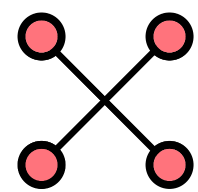

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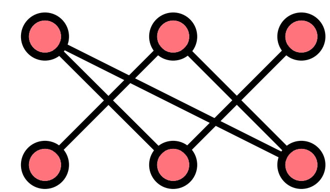


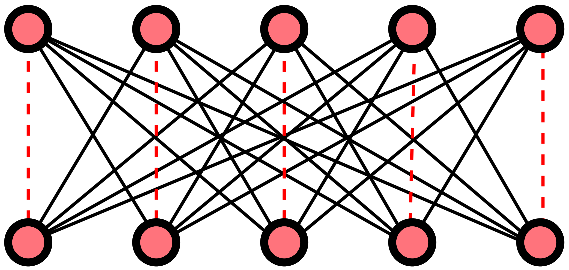
S_3

Fact For every $n \geqslant 2$, $\dim(S_n) = n$

Proof (blackboard)

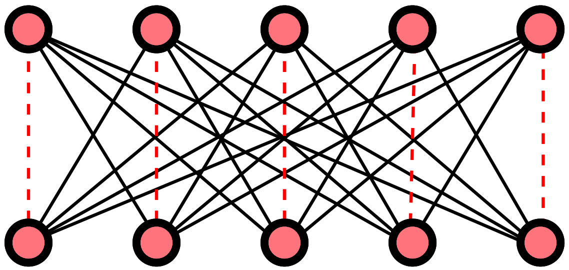
$$\dim(\mathcal{B}_n) \geqslant \dim(S_n) = n$$

Posets of height 2 have arbitrarily large dimension



Posets of height 2 have arbitrarily large dimension

Question How about the width?



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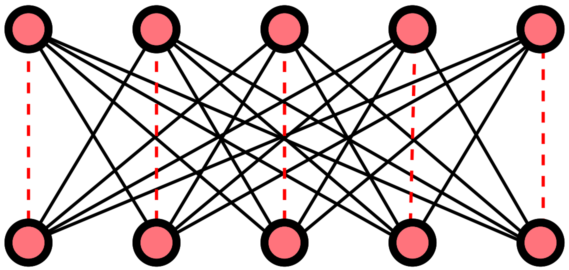

Fact (Dilworth 1950) For every poset P

$$\dim(P) \leqslant \text{width}(P)$$

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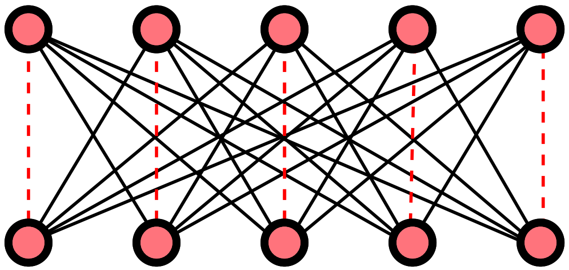


Fact (Dilworth 1950) For every poset P

$$\dim(P) \leqslant \mathrm{width}(P)$$

Proof $\mathrm{width}(P) = w$

$C_1, \ldots, C_w$ Dilworth chain partition of P

$L_1, \ldots, L_w$ linear extensions of P

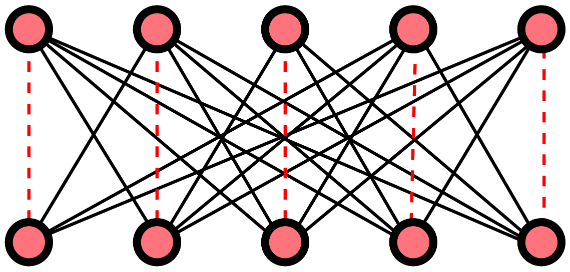
L_i produced by the generic algorithm

($\star$) prefer elements not in C_i

$$\dim(P) \leqslant \text{width}(P)$$

Large dimensional posets are wide

But not necessarilly tall



Poset's zoo

or families of posets with unbounded dimension

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Incidence posets

G graph ⇒ I_G

elements $V(G) \cup E(G)$

$$v < e \text{ in } I_G \iff \begin{array}{l} v \in V(G) \\ e \in E(G) \\ v \text{ is incident to } e \text{ in } G \end{array}$$

Poset's zoo or families of posets with unbounded dimension

Incidence posets

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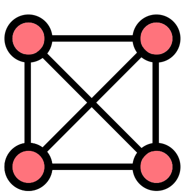
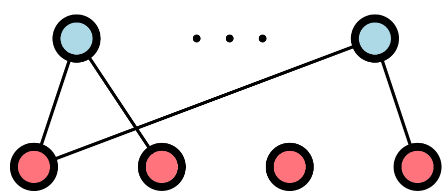
K_4 



$\binom{4}{2} = 6$ edges

4 vertices

Fact $\dim(I_{K_n}) = \Omega(\log \log n)$

Poset's zoo

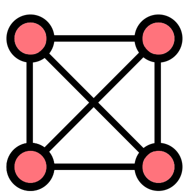
or families of posets with unbounded dimension

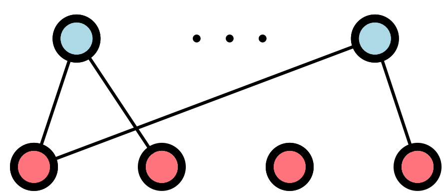
Incidence posets

G graph → I_G

elements $V(G) \cup E(G)$

$v < e$ in $I_G \iff$
$v \in V(G)$
$e \in E(G)$
v is incident to e in G

K_4 



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Fact $\dim(I_{K_n}) = \Omega(\log \log n)$ yet no large standard examples

$\mathrm{se}(I_{K_n}) \leqslant 3$

Poset's zoo or families of posets with unbounded dimension

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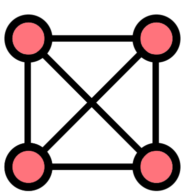
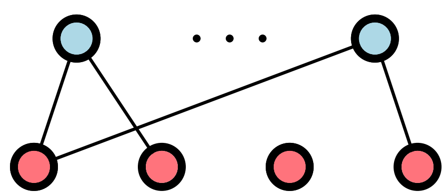
K_4 



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Proof (blackboard)

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Poset's zoo or families of posets with unbounded dimension

Interval orders

$\mathcal{I}$ family of intervals on $\mathbb{R}$

$$\forall_{I, J \in \mathcal{I}} \quad I < J \iff \forall_{x \in I,\, y \in J} \; x < y$$

Poset's zoo
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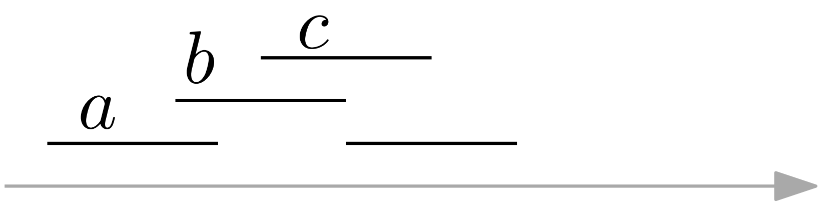
Interval orders



$a \parallel b$ but $a < c$

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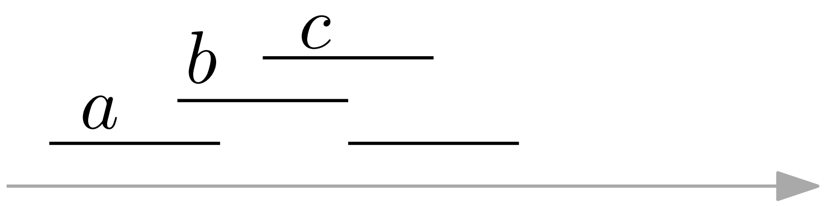

$a \parallel b$ but $a < c$

An universal interval order of order n $(n \geqslant 2)$ is the poset U_n

elements $\{[i,j] \mid i,j \in \mathbb{N},\ 1 \leqslant i < j \leqslant n\}$

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Poset's zoo or families of posets with unbounded dimension

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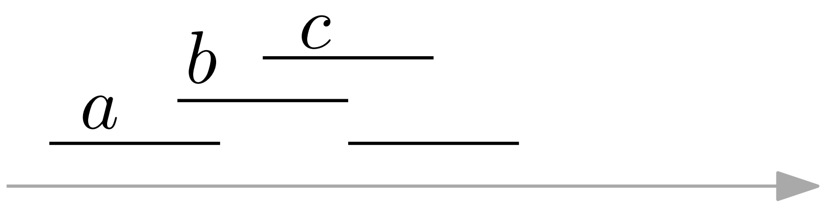

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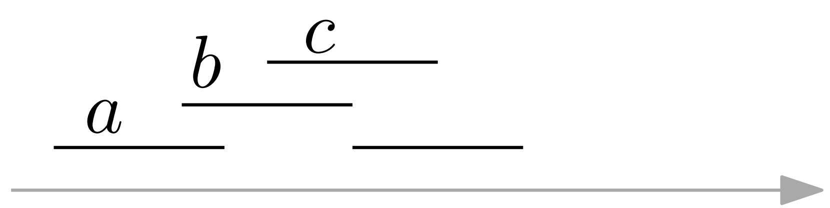

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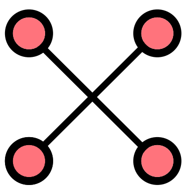
while interval orders

|||

S_2-free orders



side-tracked

The shift graph G_n of order n is the graph with

vertices	$\{(i,j) \mid i,j \in \mathbb{N},\ 1 \leqslant i < j \leqslant n\}$
edges	(i,j) adjacent with (j,k) $\forall_{1 \leqslant i < j < k \leqslant n}$

side-tracked

The shift graph G_n of order n is the graph with

vertices $\{(i,j) \mid i,j \in \mathbb{N},\ 1 \leqslant i < j \leqslant n\}$

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Fun Fact $\chi(G_n) \geqslant \lceil \log n \rceil$

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side-tracked

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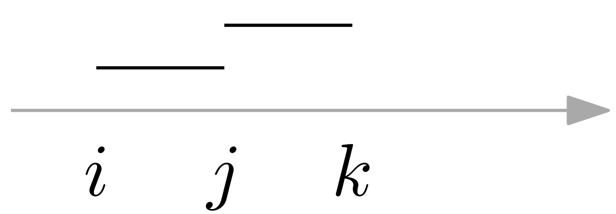
Proof $L_1, \ldots, L_d$ realizer of U_n

given (i,j,k)



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choose $\alpha \in [d]$ st

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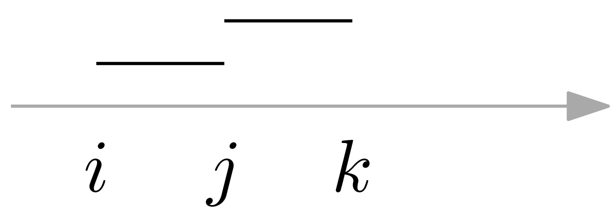
let $\varphi(i,j,k) = \alpha$



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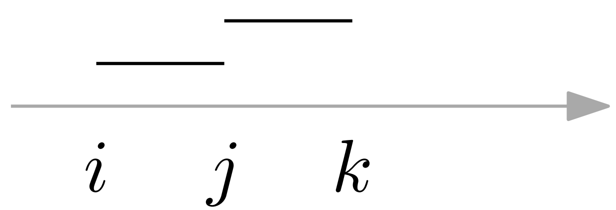

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Claim φ is a proper coloring of $G_n^{(3)}$

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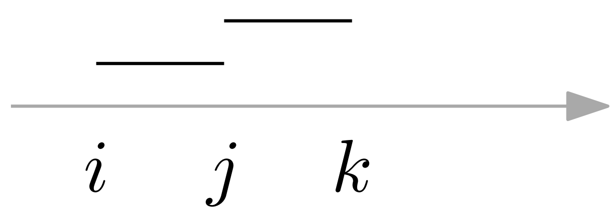

given (i,j,k)

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choose $\alpha \in [d]$ st

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thus $d \geqslant \chi(G_n^{(3)}) \geqslant \log\log n$

Claim φ is a proper coloring of $G_n^{(3)}$

Poset's zoo or families of posets with unbounded dimension

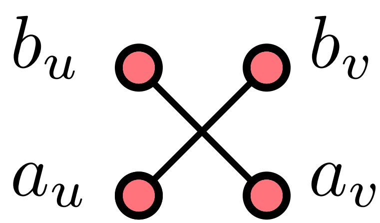
Adjacency posets

G graph $\Rightarrow$ A_G

n vertices $2n$ elements

$v \in V(G)$ $a_v \parallel b_v$ in A_G

$(u, v) \in E(G)$



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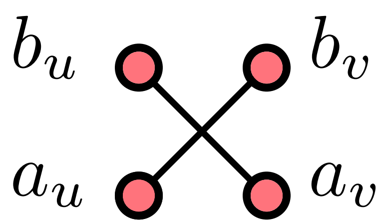

in A_G

G is triangle-free
→ $\mathrm{se}(A_G) \leqslant 2$

G has girth g
→ $\mathrm{cover}(A_G)$ has girth $\geqslant g$

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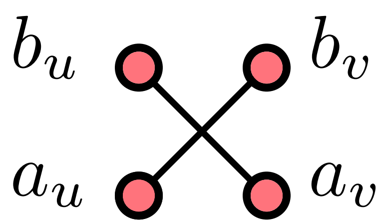

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→ $\mathrm{cover}(A_G)$ has girth $\geqslant g$

Fact $\dim(A_G) \geqslant \chi(G)$

Proof (blackboard)