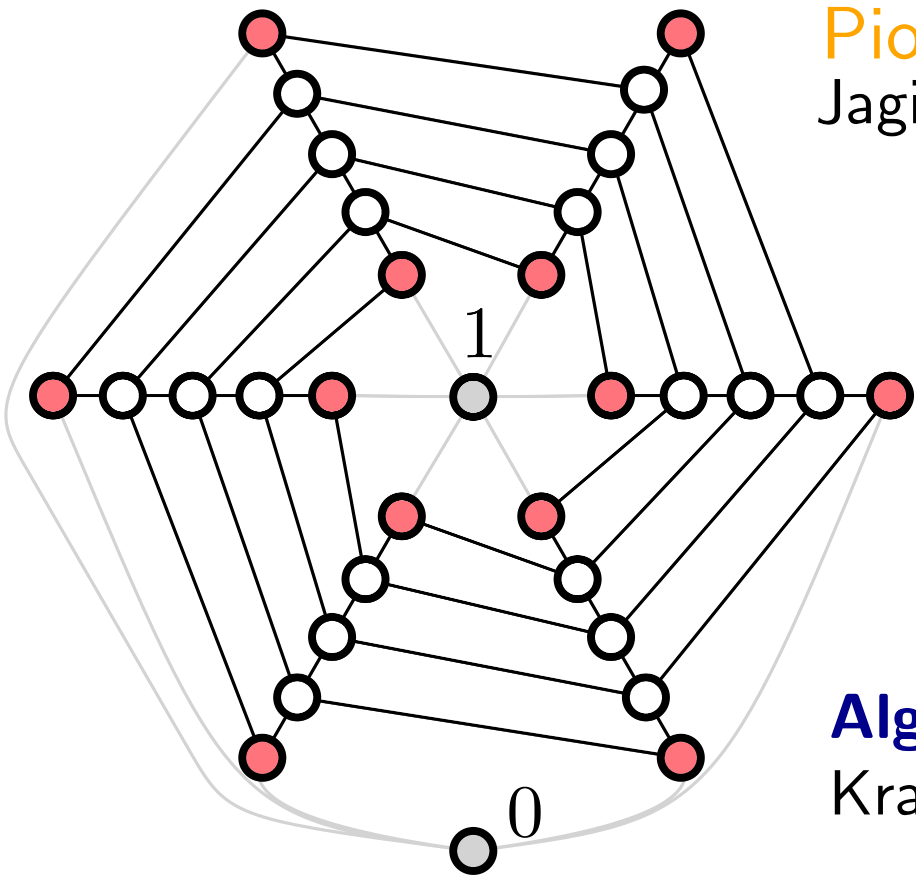
Combinatorics of posets

Lecture 2: Dimension and planarity



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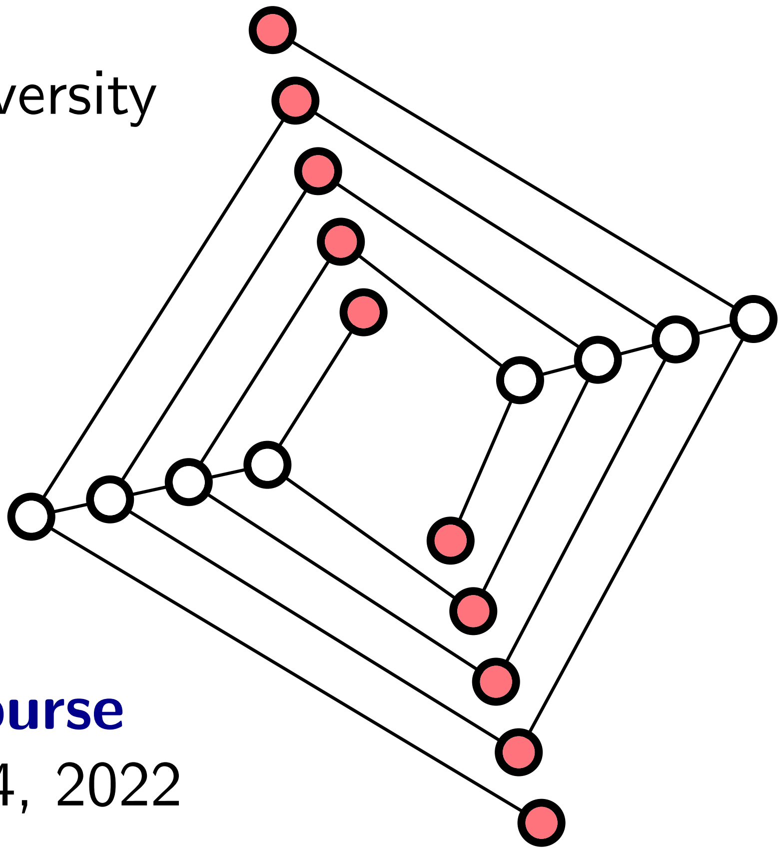

AlgoMaNet course

Kraków, May 24, 2022

$I \subseteq \mathrm{Inc}(P)$ is reversible if
there is a linear extension L of P such that
$y < x$ in L for all $(x, y) \in I$

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Question When I is reversible?

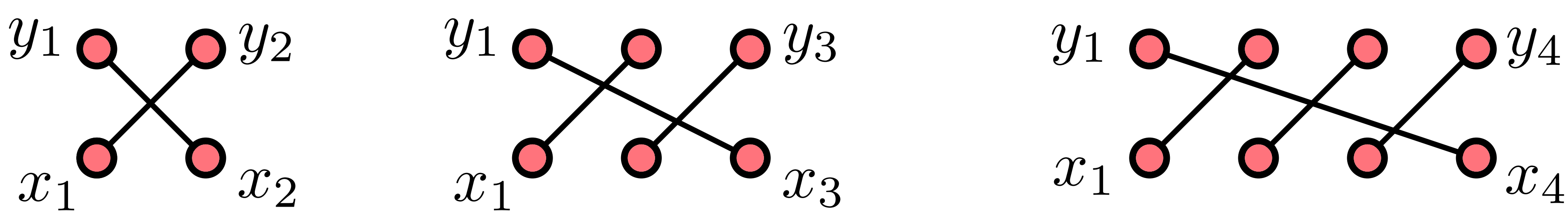
$(k \geqslant 2)$

A sequence $((x_1, y_1), \ldots, (x_k, y_k))$ of k pairs from $\mathrm{Inc}(P)$ is an alternating cycle if

$$x_i \leqslant y_{i+1} \text{ in } P \qquad \forall\, i \in [k] \quad \text{(cyclically)}$$

A sequence $((x_1, y_1), \ldots, (x_k, y_k))$ of k pairs from $\mathrm{Inc}(P)$ is a strict alternating cycle if

$$x_i \leqslant y_j \text{ in } P \iff j = i + 1 \qquad \forall\, i, j \in [k] \quad \text{(cyclically)}$$



Proposition Let P be a poset and $I \subseteq \mathrm{Inc}(P)$. Then the following conditions are equivalent

(i) I is reversible

(ii) I contains no alternating cycle

(iii) I contains no strict alternating cycle

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Proof (i) $\implies$ (ii)

suppose I is reversible
I contains an alternating cycle $((x_1, y_1), \ldots, (x_k, y_k))$
L - linear extension reversing all pairs in I

$y_i < x_i \leqslant y_{i+1}$ in L $\forall_{\, i \in [k]}$ (cyclically)

Proposition Let P be a poset and $I \subseteq \mathrm{Inc}(P)$.
Then the following conditions are equivalent

(i) I is reversible

(ii) I contains no alternating cycle

(iii) I contains no strict alternating cycle

Proof (ii) $\Longrightarrow$ (i)

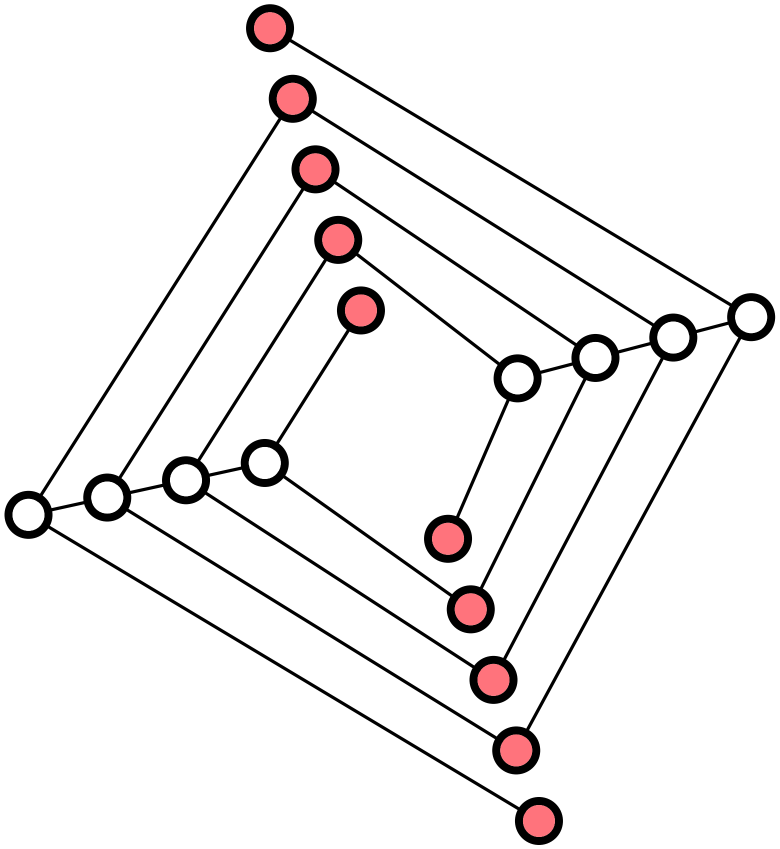
Claim $P \cup I^d$ is acyclic
Proof (blackboard)

L linear extension of $P \cup I^d$

→ L is a linear extension of P

→ L reverses all pairs in I

Kelly's example (1981)



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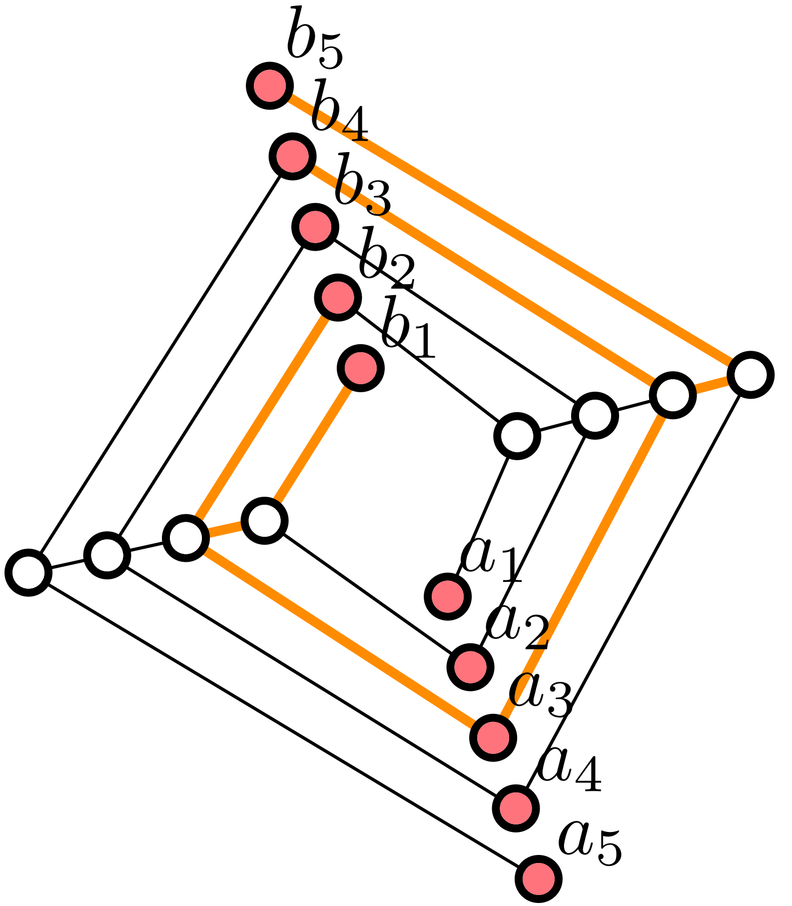

1. $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ induce S_n
2. thus $\dim(Q_n) \geqslant n$
3. $\mathrm{tw}(\mathrm{cover}(Q_n)) \leqslant \mathrm{pw}(\mathrm{cover}(Q_n)) \leqslant 3$

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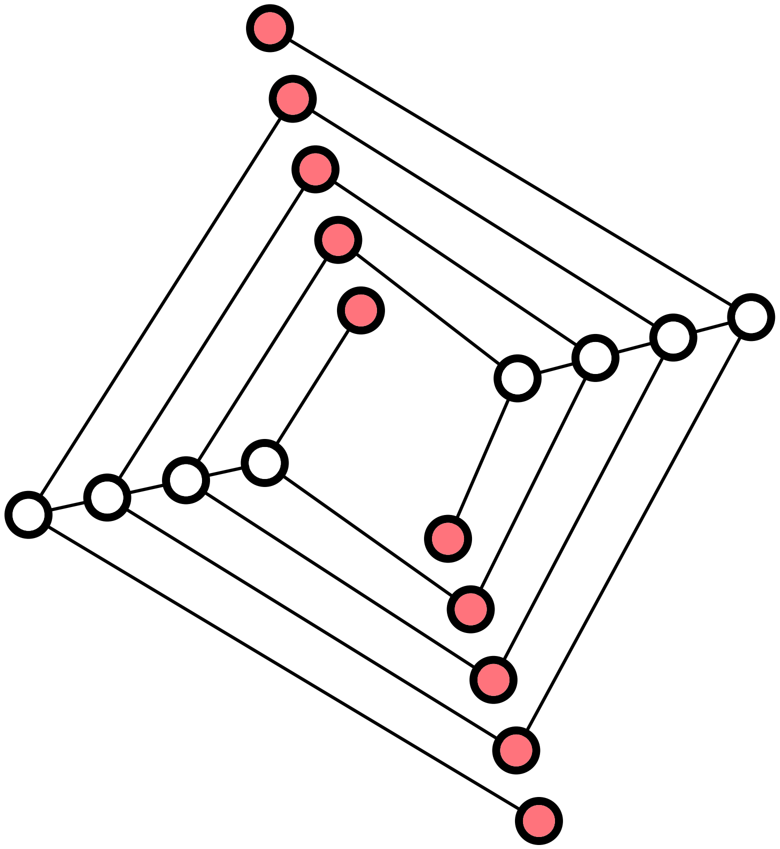

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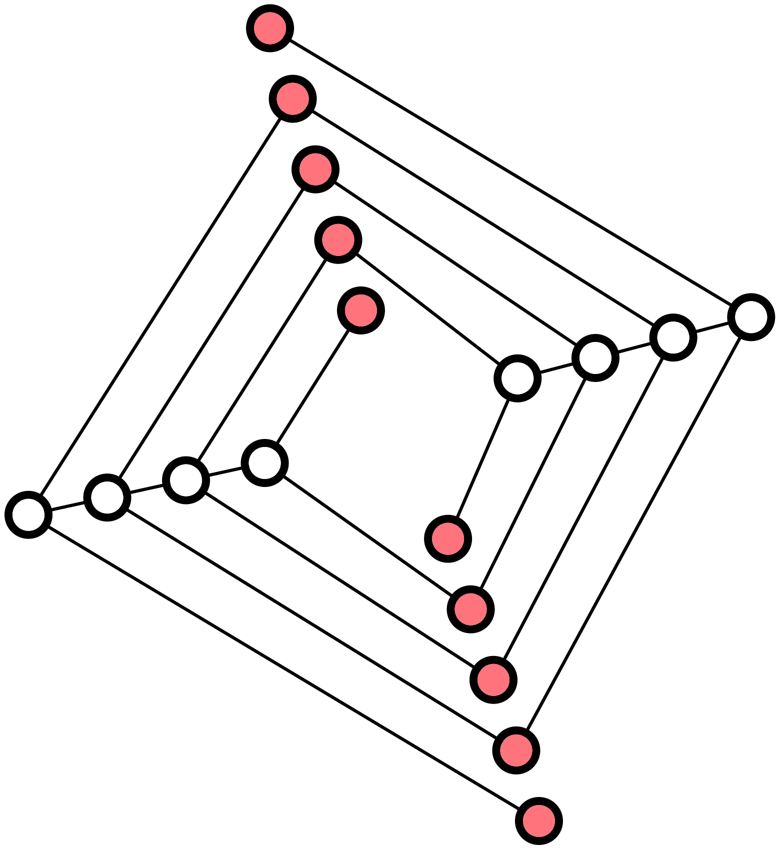
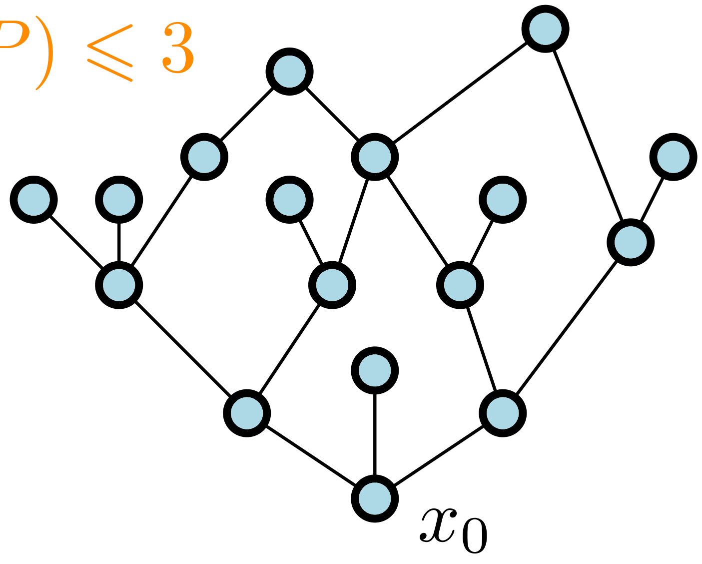


1 $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ induce S_n

2 thus $\dim(Q_n) \geqslant n$

3 $\mathrm{tw}(\mathrm{cover}(Q_n)) \leqslant \mathrm{pw}(\mathrm{cover}(Q_n)) \leqslant 3$

Theorem (Trotter, Moore 1977)
Let P be a poset with a planar diagram that has a unique minimal element. Then $\dim(P) \leqslant 3$



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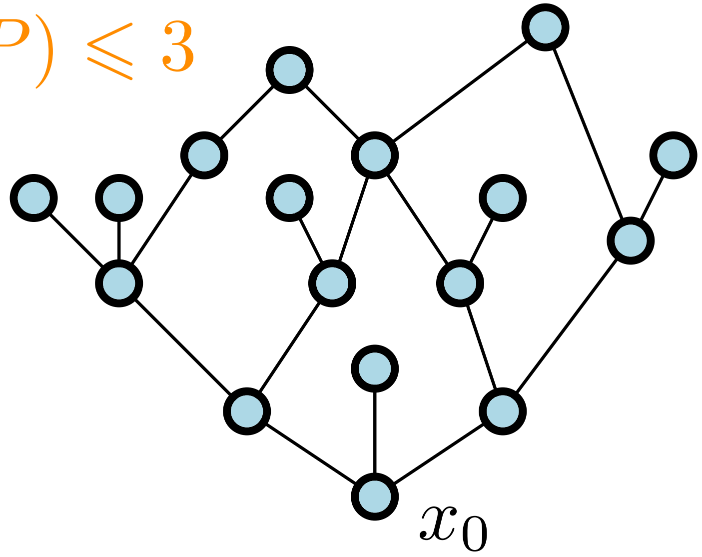
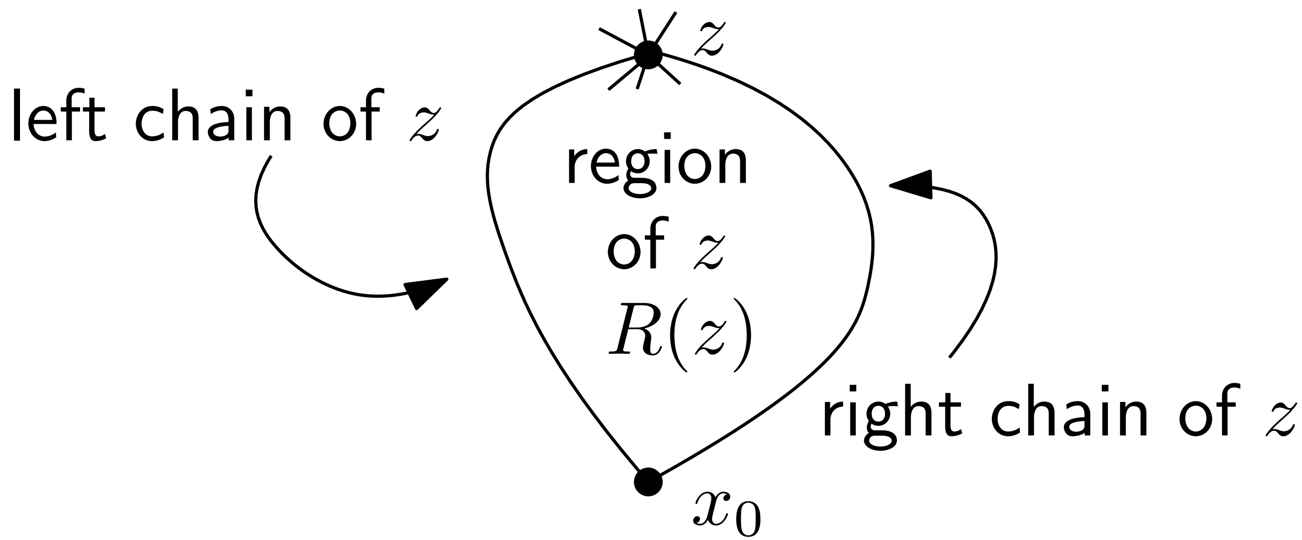
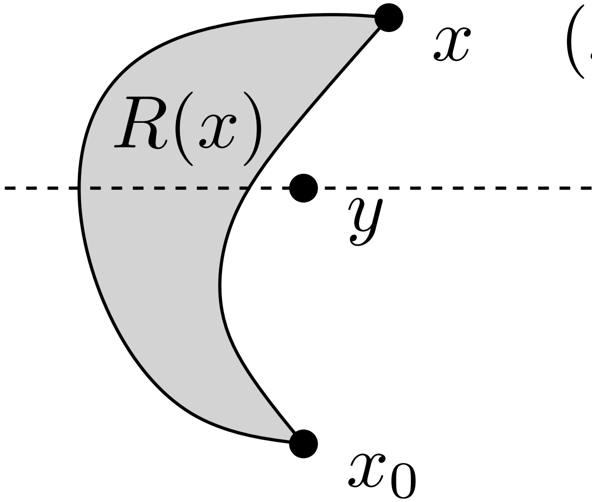


Proof Fix a planar diagram of P



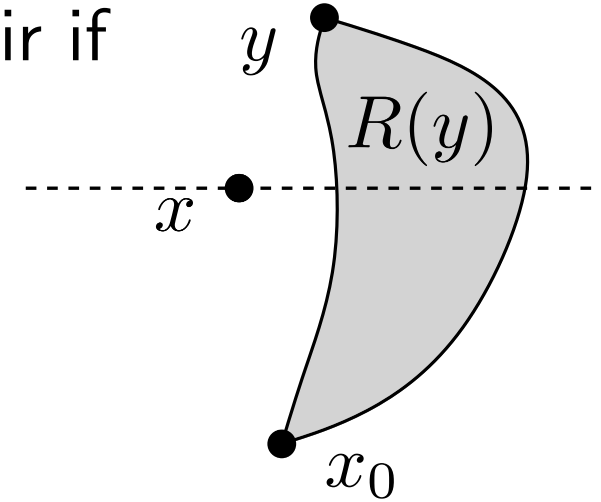
let $(x, y) \in \mathrm{Inc}(P)$

(x, y) is a left pair if



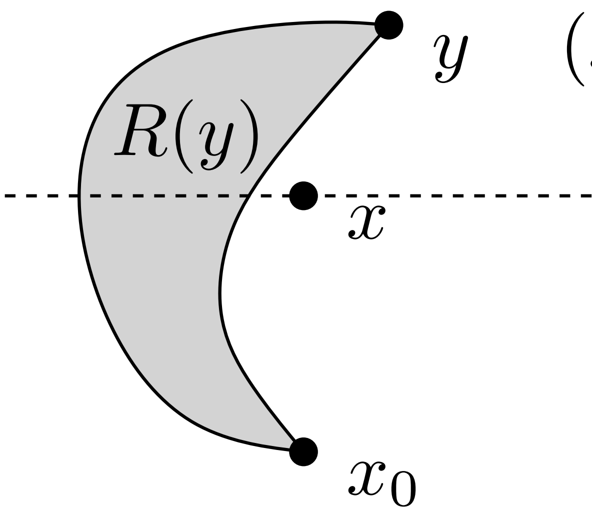
$R(x)$ is left of y at the y-line

or



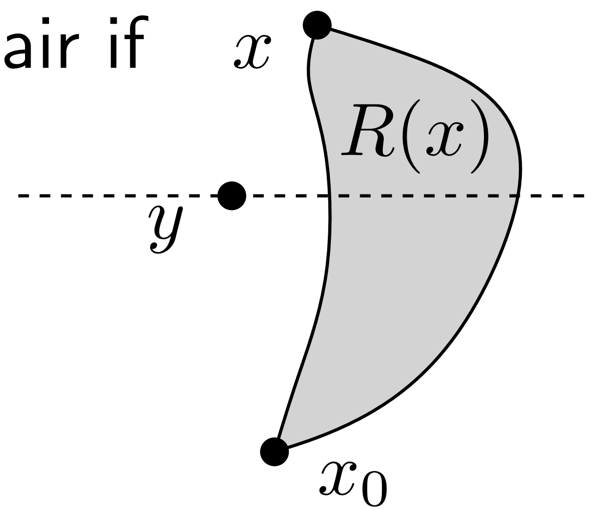
x is left of $R(y)$ at the x-line

(x, y) is a right pair if



x is right of $R(y)$ at the x-line

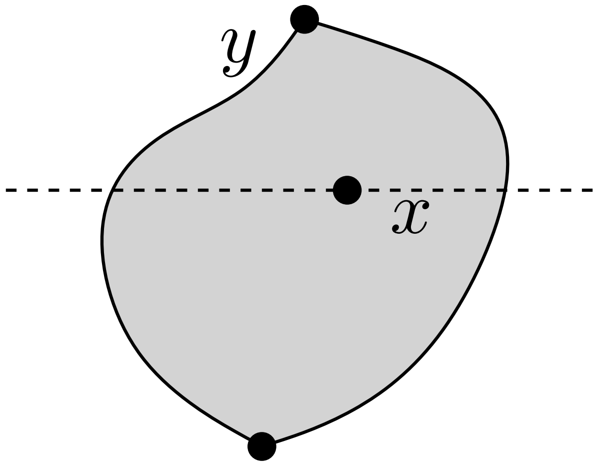
or



$R(x)$ is right of y at the y-line

let $(x, y) \in \mathrm{Inc}(P)$

(x, y) is a under pair if $R(x) \subsetneq R(y)$



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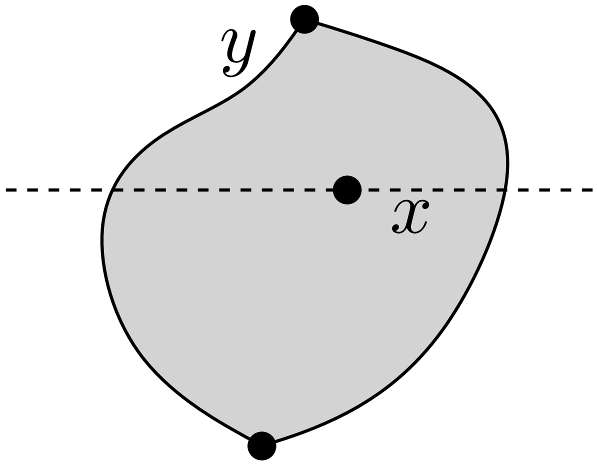

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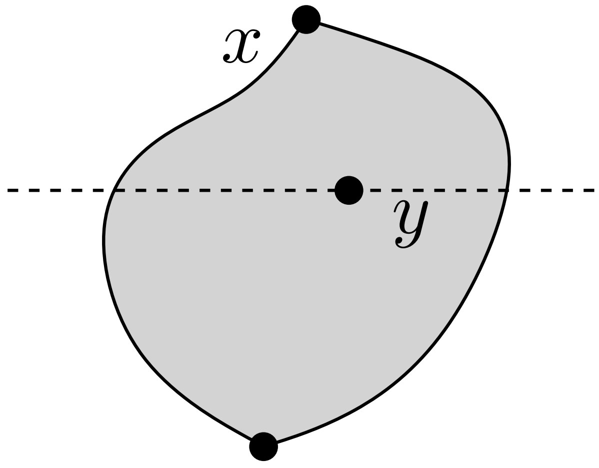

$$\mathrm{Inc}(P) = I_{\mathsf{left}} \cup I_{\mathsf{right}} \cup I_{\mathsf{under}} \cup I_{\mathsf{over}}$$

$$I_{\mathsf{right}} = I^{d}_{\mathsf{left}}$$

$$I_{\mathsf{over}} = I^{d}_{\mathsf{under}}$$

$$\mathrm{Inc}(P) = I_{\mathsf{left}} \cup I_{\mathsf{right}} \cup I_{\mathsf{under}} \cup I_{\mathsf{over}}$$

$$I_{\mathsf{right}} = I^d_{\mathsf{left}}$$

$$I_{\mathsf{over}} = I^d_{\mathsf{under}}$$

Claim I_{over} is reversible
Proof (just take L given by the increasing y-coordinates)

$$\mathrm{Inc}(P) = I_{\mathsf{left}} \cup I_{\mathsf{right}} \cup I_{\mathsf{under}} \cup I_{\mathsf{over}}$$

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Claim I_{over} is reversible
Proof (just take L given by the increasing y-coordinates)

Claim I_{left} is reversible
Proof (blackboard)

$$\mathrm{Inc}(P) = I_{\mathsf{left}} \cup I_{\mathsf{right}} \cup I_{\mathsf{under}} \cup I_{\mathsf{over}}$$

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Claim $I_{\mathsf{left}} \cup I_{\mathsf{over}}$ is reversible
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$$\mathrm{Inc}(P) = I_{\mathsf{left}} \cup I_{\mathsf{right}} \cup I_{\mathsf{under}} \cup I_{\mathsf{over}}$$

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Claim I_{under} is reversible
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$\mathrm{Inc}(P) = I_{\text{left}} \cup I_{\text{right}} \cup I_{\text{under}} \cup I_{\text{over}}$

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Proof (blackboard)

Claim I_{under} is reversible
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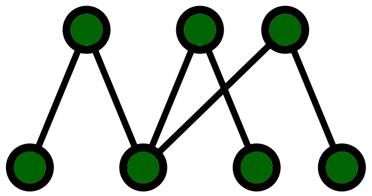
so $\mathrm{Inc}(P)$ can be covered three reversible sets:

$I_{\text{left}} \cup I_{\text{over}}$
$I_{\text{right}} \cup I_{\text{over}}$
I_{under}

so $\dim(P) \leqslant 3$

Theorem (Trotter, Moore 1977)
Let P be a poset with a planar diagram that has a unique minimal element. Then $\dim(P) \leqslant 3$

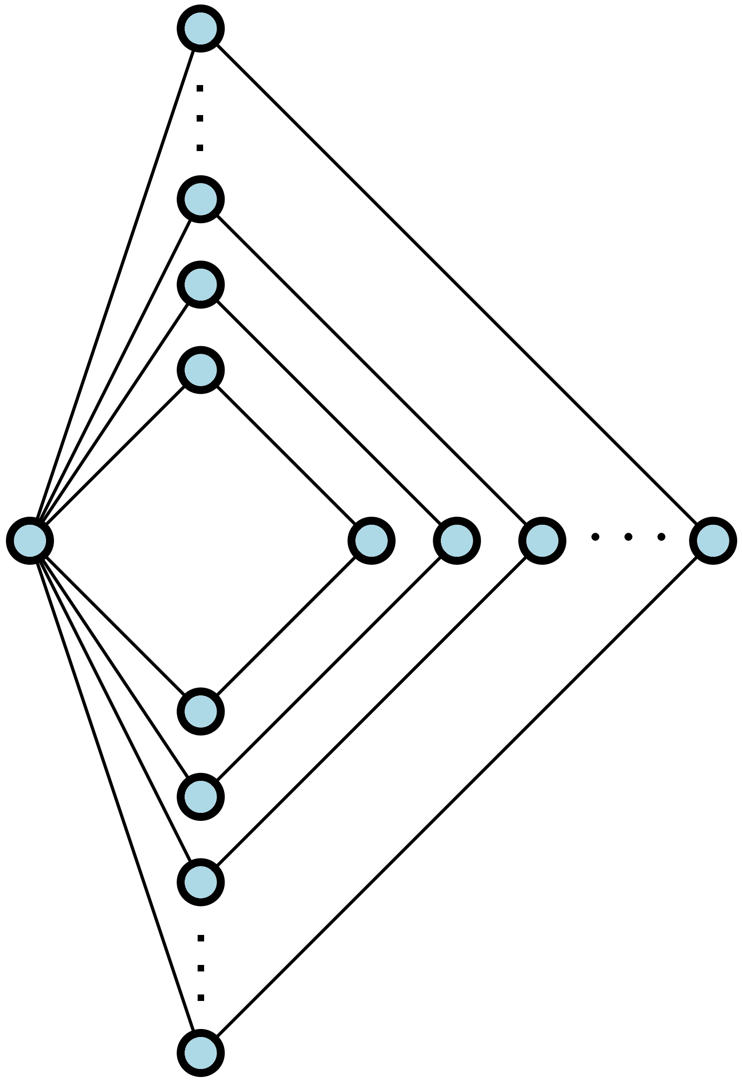
Corollary Let P be a poset whose cover graph is a forest. Then $\dim(P) \leqslant 3$



Proof (blackboard)

Theorem (Felsner, Trotter, Wiechert 2015)
Let P be a poset with an outerplanar cover graph. Then

$$\dim(P) \leqslant 4$$



Theorem (Felsner, Trotter, Wiechert 2015)

Let P be a poset with an outerplanar cover graph. Then

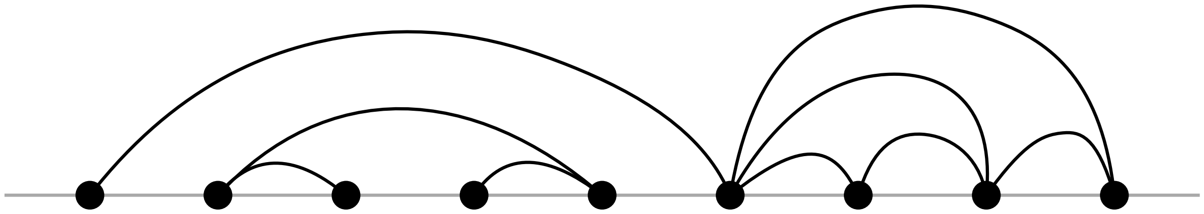
$$\dim(P) \leqslant 4$$

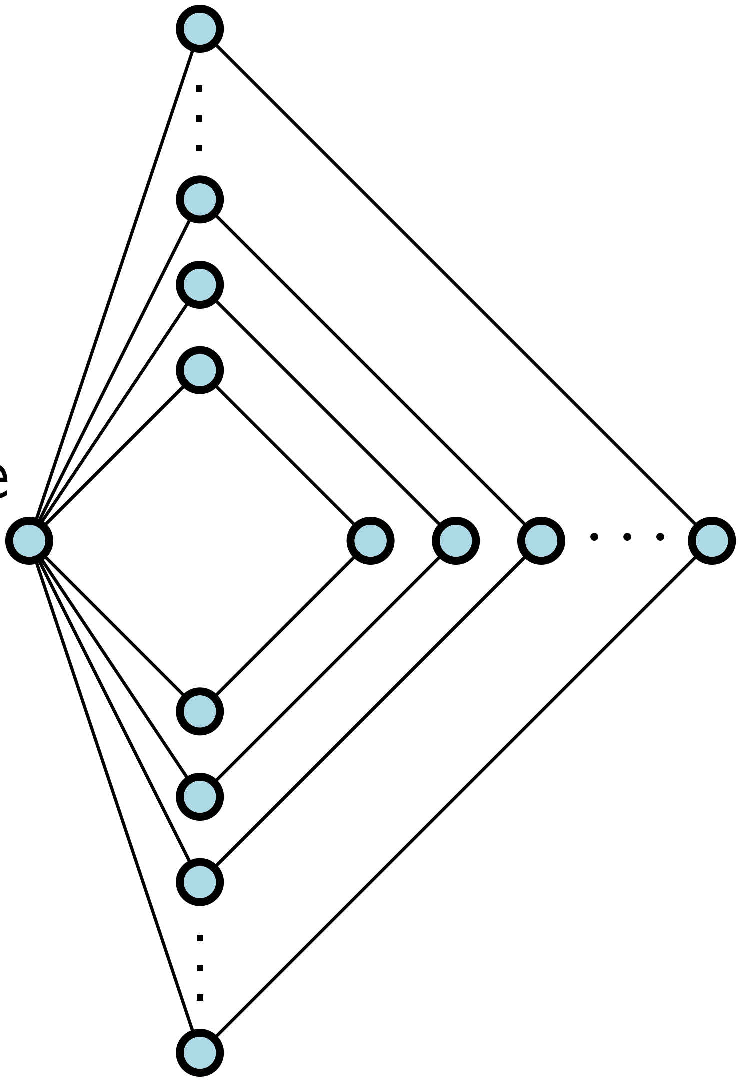
Proof

let G be the covergraph of P

fix a plane embedding of G such that

- all vertices lie on a single line
- all edges are in the upper halfplane





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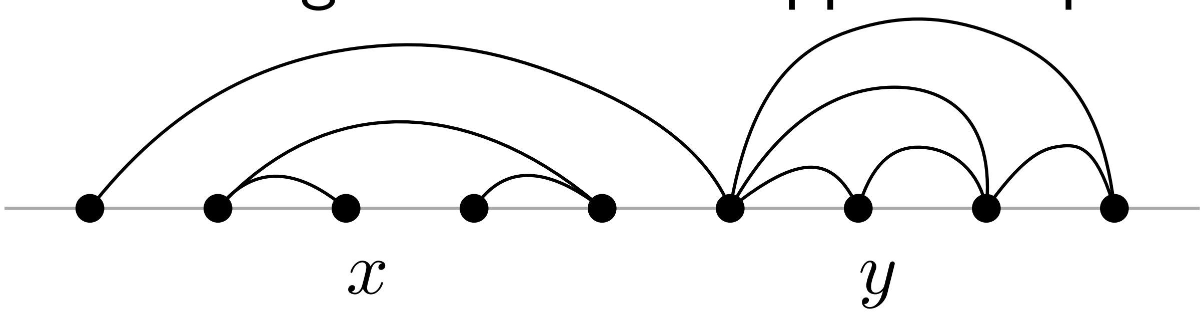

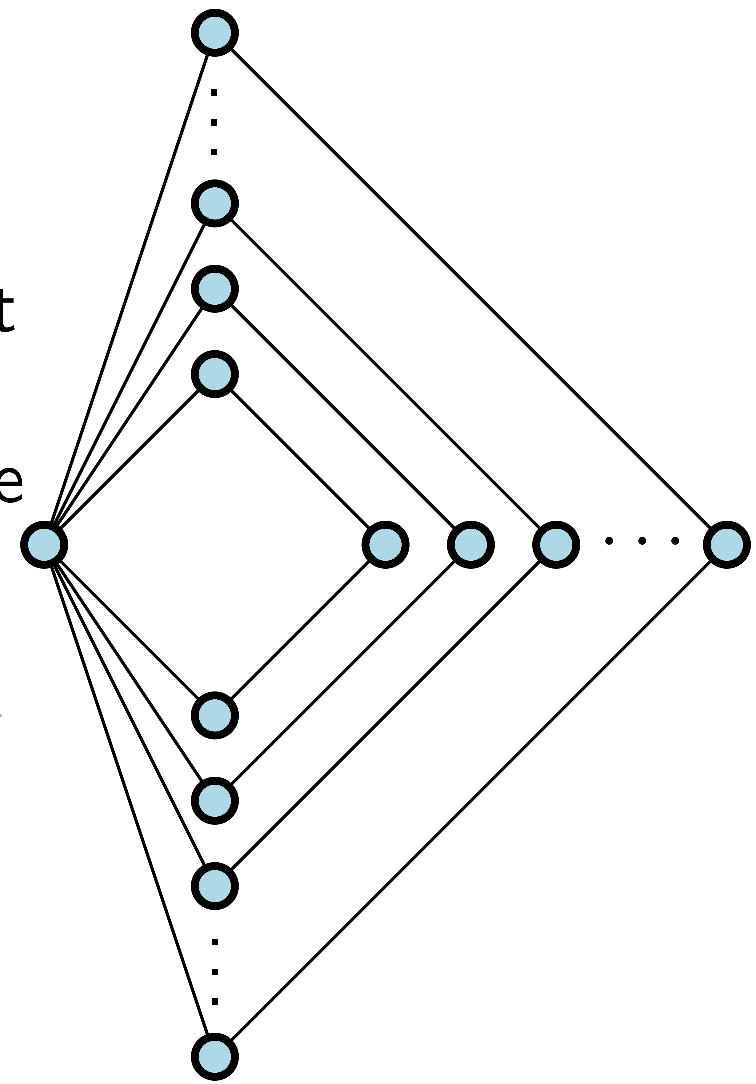

let $(x, y) \in \mathrm{Inc}(P)$

x is left of y

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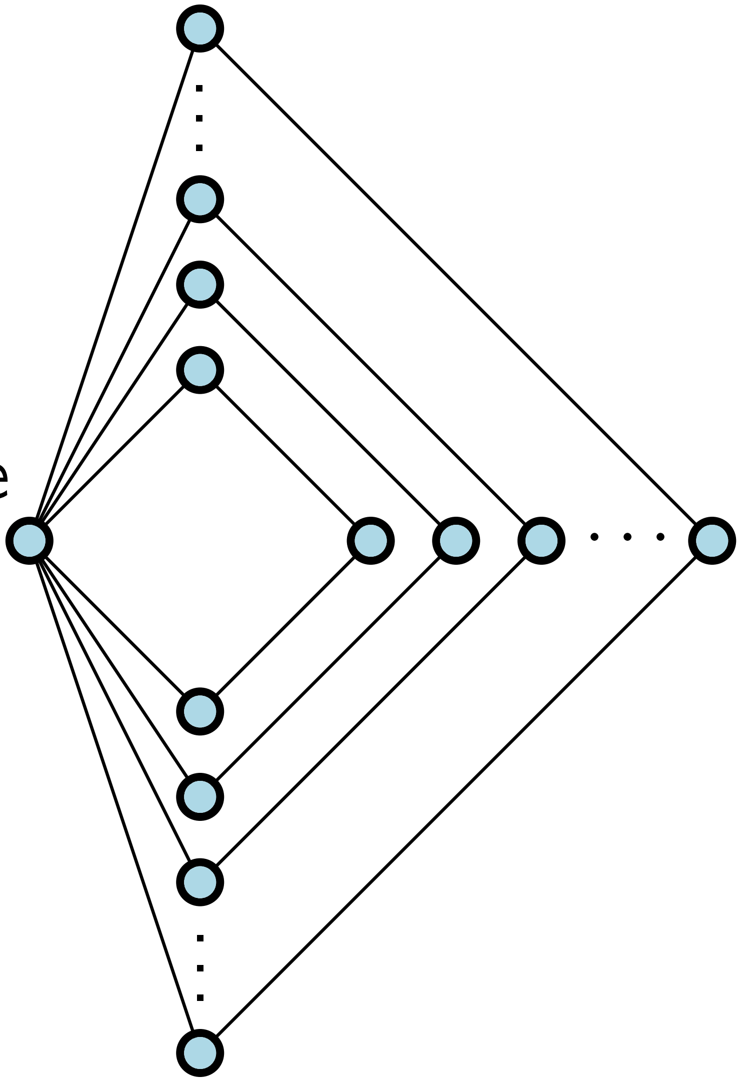

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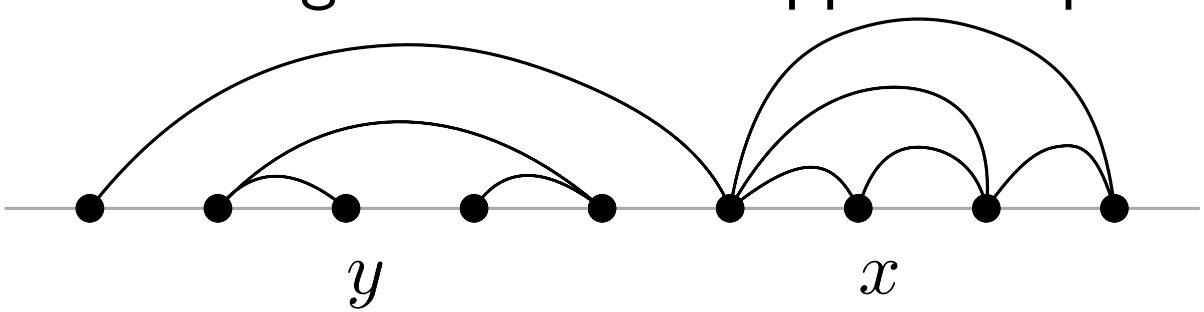

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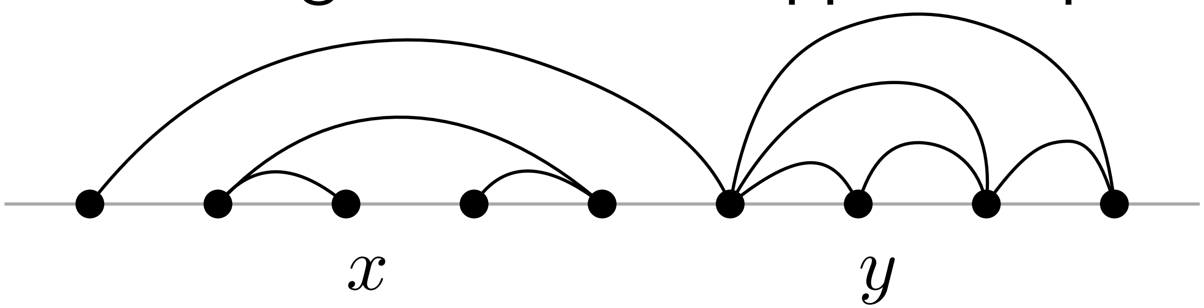
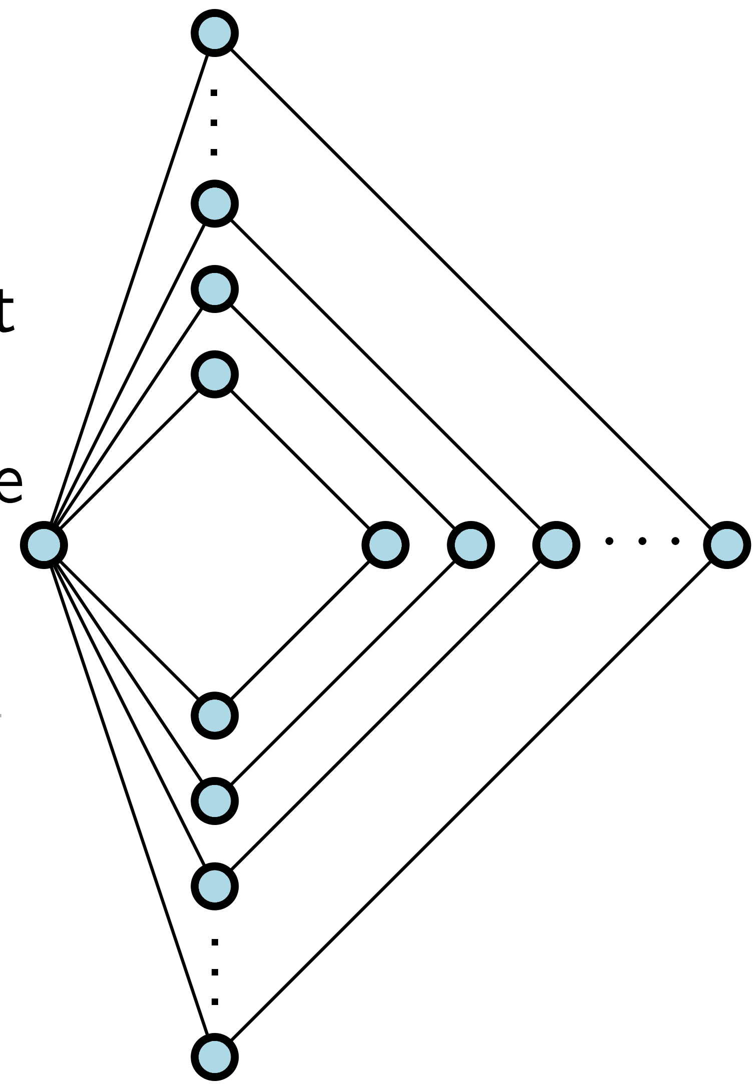


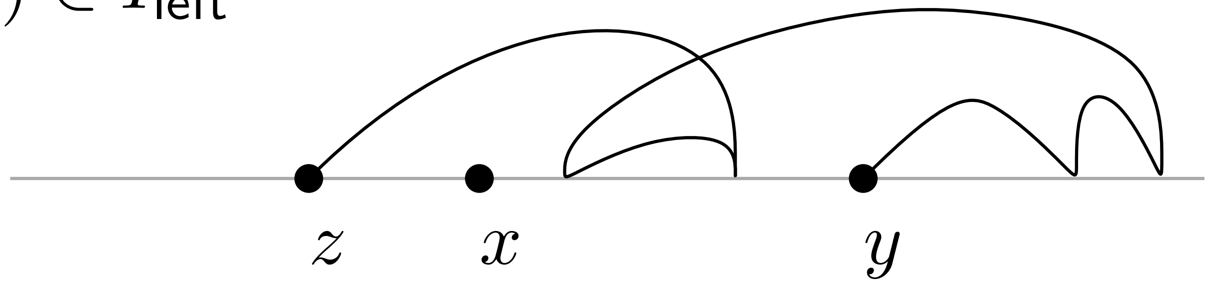


let $(x, y) \in \mathrm{Inc}(P)$

x is left of y

$\mathrm{Inc}(P) = I_{\text{left}} \cup I_{\text{right}}$

let $(x, y) \in I_{\text{left}}$



(x, y) is active if $\exists_z$ such that $z < y$ in P and z is left of x

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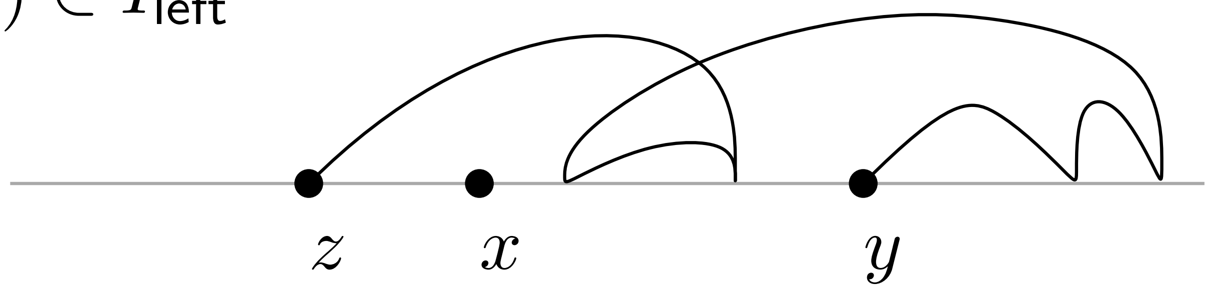
let $(x, y) \in I_{\mathsf{left}}$



(x, y) is active if $\exists_z$ such that $z < y$ in P and z is left of x

$I_{\mathsf{left}} = I_{\mathsf{left}}^{\mathsf{active}} \cup I_{\mathsf{left}}^{\mathsf{non\text{-}active}}$

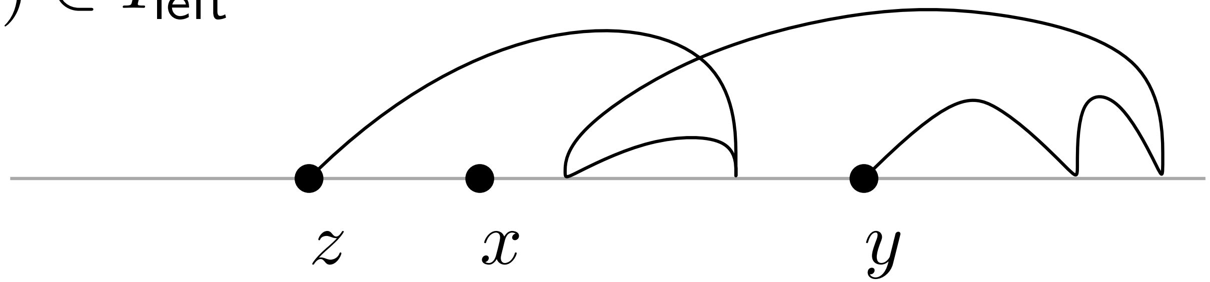
Claim $I_{\mathsf{left}}^{\mathsf{active}}$ is reversible

Claim $I_{\mathsf{left}}^{\mathsf{non\text{-}active}}$ is reversible

Proofs (blackboard)

$\mathrm{Inc}(P) = I_{\text{left}} \cup I_{\text{right}}$

let $(x, y) \in I_{\text{left}}$



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Claim $I_{\text{left}}^{\text{active}}$ is reversible

Claim $I_{\text{left}}^{\text{non-active}}$ is reversible

Proofs (blackboard)

so $\mathrm{Inc}(P)$ can be covered four reversible sets:

$I_{\text{left}}^{\text{active}}$ $I_{\text{left}}^{\text{non-active}}$

$I_{\text{right}}^{\text{active}}$ $I_{\text{right}}^{\text{non-active}}$

so $\dim(P) \leqslant 4$

Theorem (Streib, Trotter 2014)
There is a function $f : \mathbb{N} \to \mathbb{N}$ such that for every $h \geqslant 1$ and every poset P of height $\leqslant h$ and with a planar cover graph, we have

$$\dim(P) \leqslant f(h)$$

Large dimensional posets are wide

Large dimensional planar posets are wide

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Large dimensional posets are wide

Large dimensional planar posets are wide

Theorem (Joret, PM, Ossona de Mendez, Wiechert 2019)
Let P be a poset of height at most h with a cover graph G such that $\mathrm{wcol}_{3h-3}(G) \leqslant c$. Then

$$\dim(P) \leqslant 4^c$$

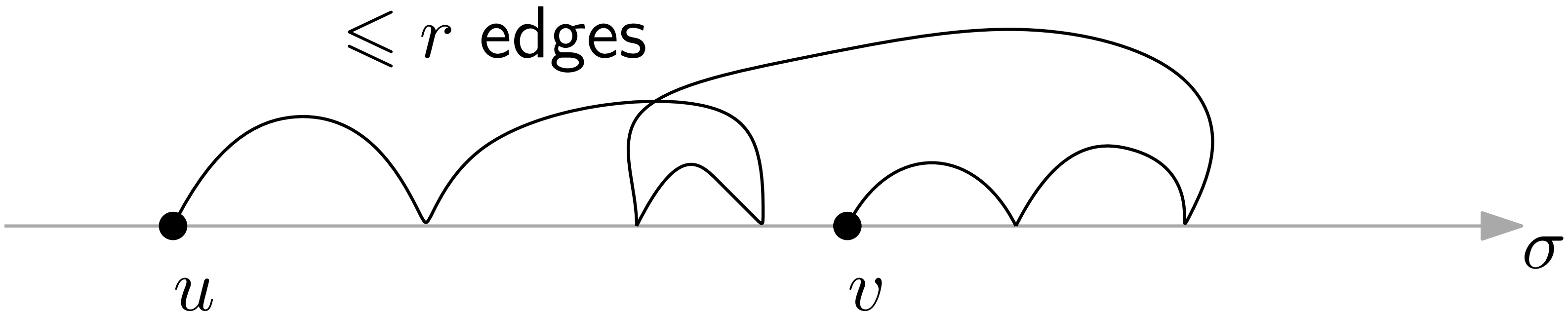
G graph

σ linear order on $V(G)$

$r \in \mathbb{N}$

$u, v \in V(G)$

u is r-weakly reachable from v w.r.t. σ in G if
there is a path Q of length $\leqslant r$ from v to u in G such that
$$\min_\sigma(Q) = u$$



wrt. σ in G

$$\mathrm{WReach}_r[G, \sigma, v] = \{u \mid u \text{ is } r\text{-weakly reachable from } v\}$$

$$\mathrm{wcol}_r(G) = \min_\sigma \max_{v \in V(G)} |\mathrm{WReach}_r[G, \sigma, v]|$$

introduced in 2004 by Kierstead and Yang

G graph
σ linear order on $V(G)$
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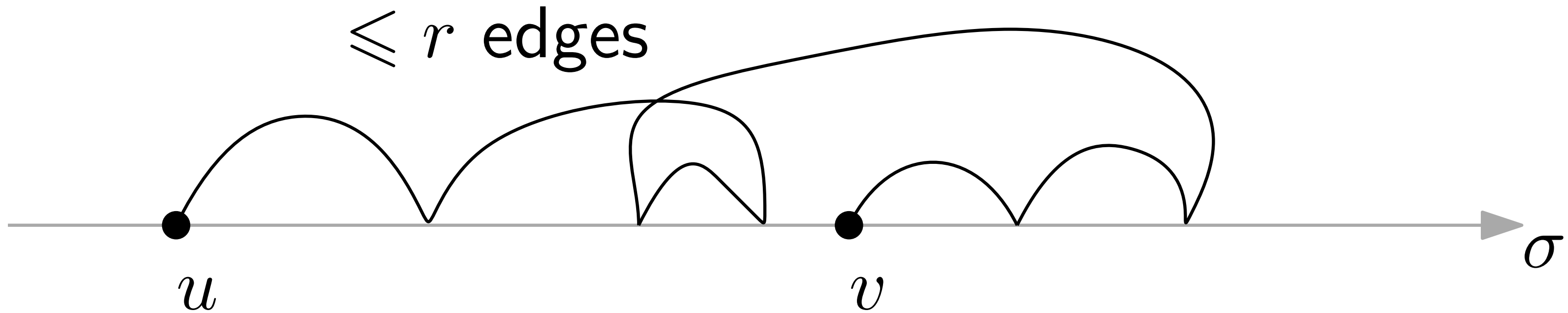

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$\mathrm{wcol}_r(G) = \min_\sigma \max_{v \in V(G)} |\mathrm{WReach}_r[G, \sigma, v]|$

(Van den Heuvel, Ossona de Mendez, Quiroz, Rabinovich, Siebertz 2017)

G planar $\quad \mathrm{wcol}_r(G) = \mathcal{O}(r^3)$

and there is a family of planar graphs G with

$$\mathrm{wcol}_r(G) = \Omega(r^2 \log r)$$

(Joret, PM 2022)

Theorem

Let P be a poset of height at most h with a cover graph G such that $\mathrm{wcol}_{3h-3}(G) \leqslant c$. Then

$$\dim(P) \leqslant 4^c$$

Proof (blackboard)