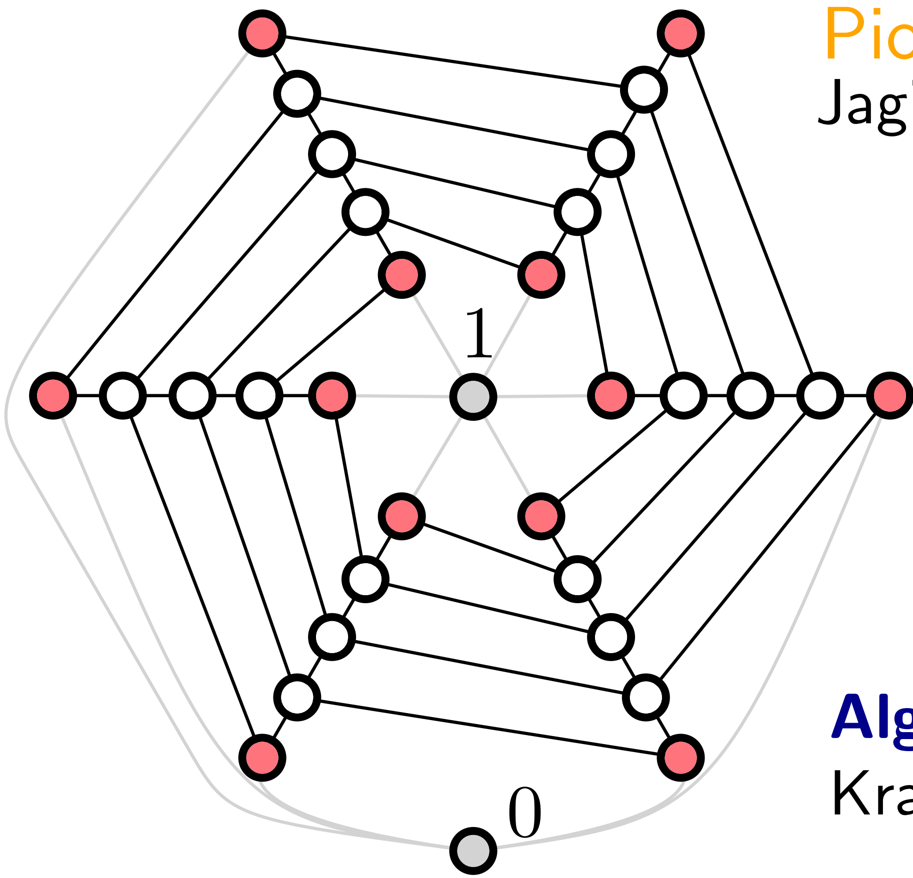
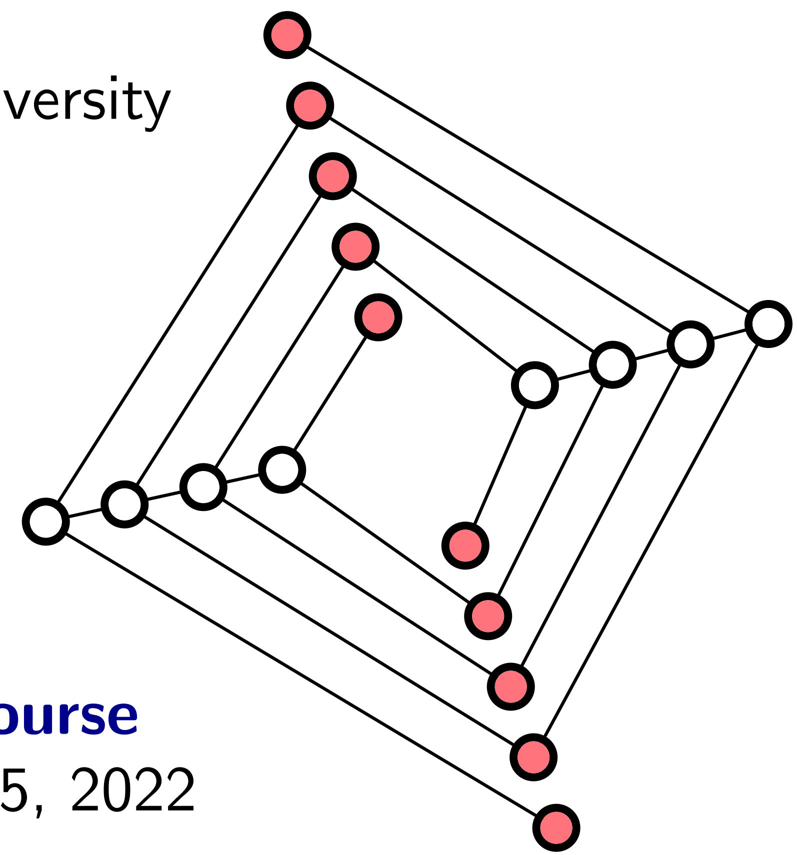
Combinatorics of posets

Lecture 3: Dimension and sparsity



Piotr Micek

Jagiellonian University



AlgoMaNet course

Kraków, May 25, 2022

Theorem

Let P be a poset of height at most h with a cover graph G such that $\mathrm{wcol}_{3h-3}(G) \leqslant c$. Then

$$\dim(P) \leqslant 4^c$$

Proof (blackboard)

Question What graphs do we need to forbid as a minor in a cover graph to have bounded dimension?

Rephrasing What graphs are unavoidable as minors in cover graphs of posets of large dimension?

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Rephrasing What graphs are unavoidable as minors in cover graphs of posets of large dimension?

cover graph of P	
forest	$\dim(P) \leqslant 3$
outerplanar	$\dim(P) \leqslant 4$
treewidth $\leqslant 2$	$\dim(P) \leqslant 12$
planar treewidth$= 3$	∞

Question What graphs do we need to forbid as a minor in a cover graph to have bounded dimension?

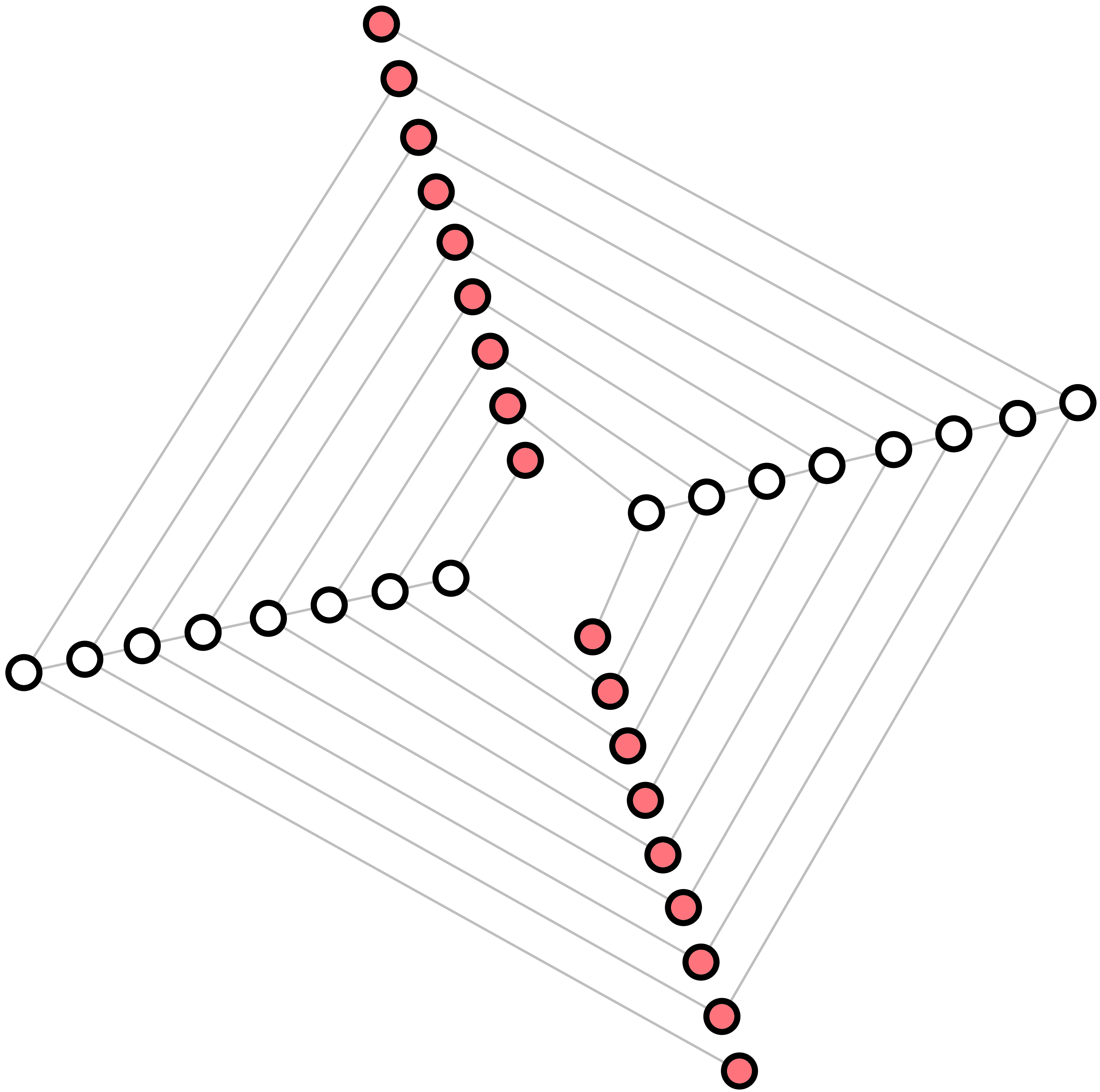
Rephrasing What graphs are unavoidable as minors in cover graphs of posets of large dimension?

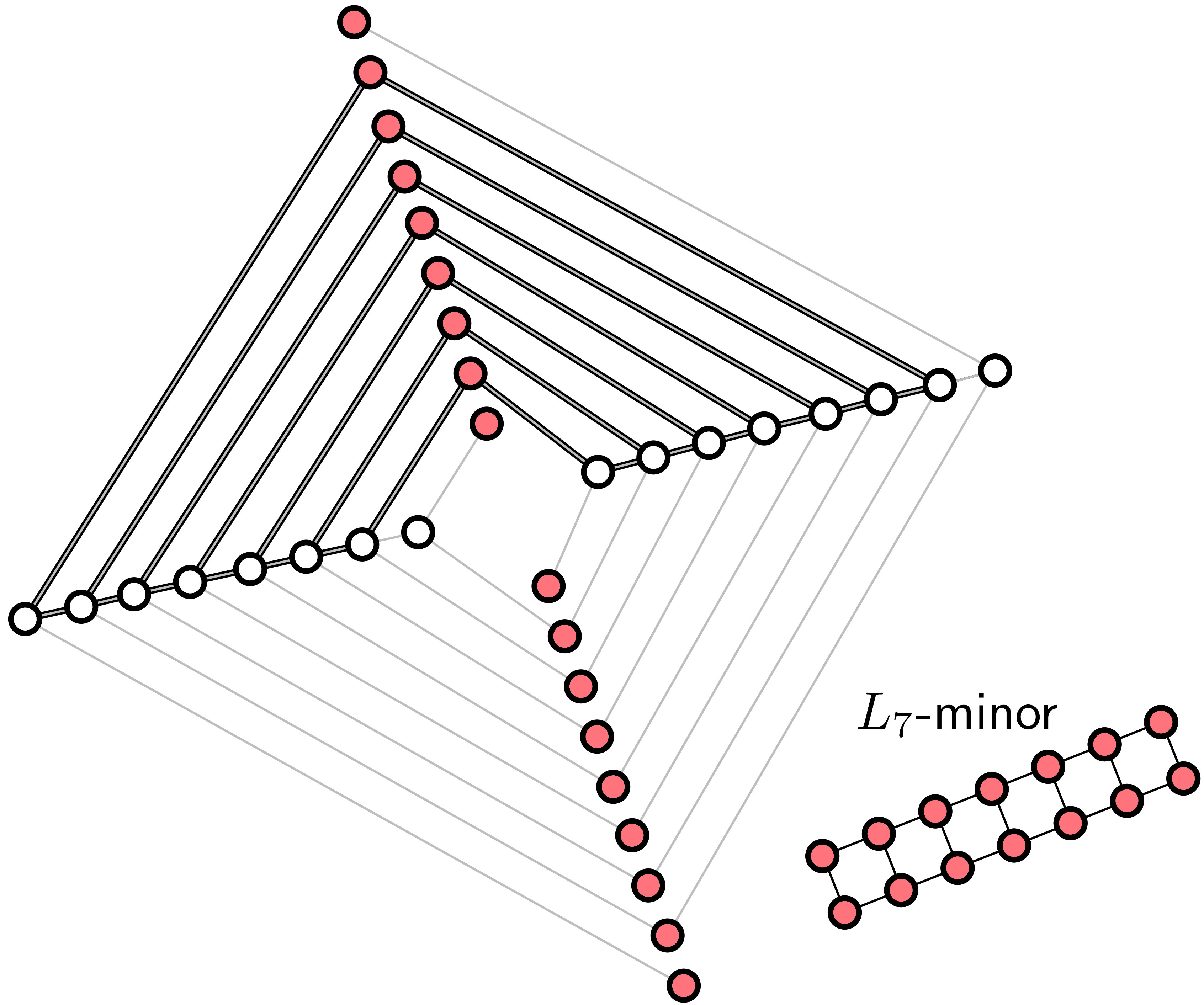
cover graph of P	
forest	$\dim(P) \leqslant 3$ (Trotter, Moore 1977)
outerplanar	$\dim(P) \leqslant 4$ (Felsner, Trotter, Wiechert 2015)
treewidth $\leqslant 2$	$\dim(P) \leqslant 12$ (Seweryn 2020)
planar treewidth$= 3$	∞

Question What graphs do we need to forbid as a minor in a cover graph to have bounded dimension?

Rephrasing What graphs are unavoidable as minors in cover graphs of posets of large dimension?

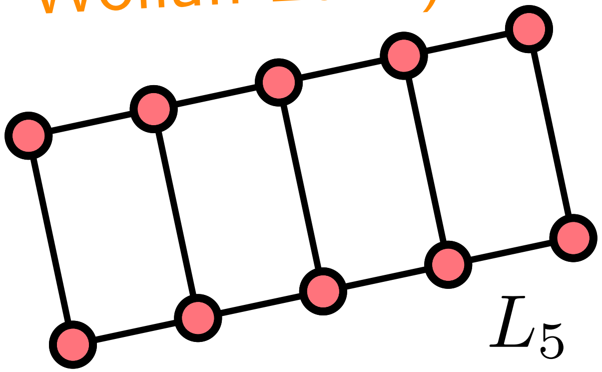
	cover graph of P	
K_3	forest	$\dim(P) \leqslant 3$
	outerplanar	$\dim(P) \leqslant 4$
K_4	treewidth $\leqslant 2$	$\dim(P) \leqslant 12$
K_5	planar treewidth $= 3$	∞




L_7-minor

cover graph of P	
forest	$\dim(P) \leqslant 3$
outerplanar	$\dim(P) \leqslant 4$
treewidth $\leqslant 2$	$\dim(P) \leqslant 12$
no L_k minor	$\dim(P) \leqslant f(k)$

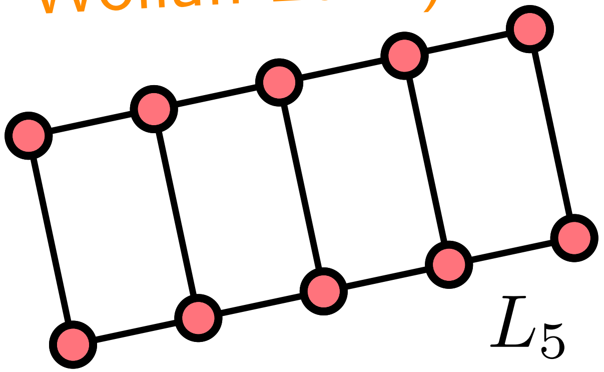
(Huynh, Joret, PM, Seweryn, Wollan 2020)



L_5

cover graph of P	
forest	$\dim(P) \leqslant 3$
outerplanar	$\dim(P) \leqslant 4$
treewidth $\leqslant 2$	$\dim(P) \leqslant 12$
no L_k minor	$\dim(P) \leqslant f(k)$

(Huynh, Joret, PM, Seweryn, Wollan 2020)



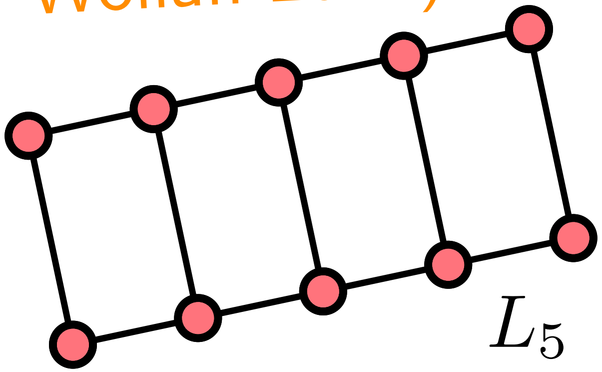
Conjecture
H is unavoidable $\Leftrightarrow$
H is a minor of the cover graph of a Kelly example

cover graph of P

forest	$\dim(P) \leqslant 3$
outerplanar	$\dim(P) \leqslant 4$
treewidth $\leqslant 2$	$\dim(P) \leqslant 12$
pathwidth $\leqslant 2$	$\dim(P) \leqslant 6$
no L_k minor	$\dim(P) \leqslant f(k)$

(Wiechert 2017)

(Huynh, Joret, PM, Seweryn, Wollan 2020)



Conjecture

H is unavoidable $\Leftrightarrow$
H is a minor of the cover graph of a Kelly example

height and dimension

Theorem (Streib, Trotter 2014)

There is a function $f : \mathbb{N} \to \mathbb{N}$ such that for every $h \geqslant 1$ and every poset P of height $\leqslant h$ and with a planar cover graph, we have

$$\dim(P) \leqslant f(h)$$

Large dimensional posets are wide

Large dimensional planar posets are tall

height and dimension

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Large dimensional posets are wide

Large dimensional *planar* *posets are* tall

(Joret, PM, Ossona de Mendez, Wiechert 2019)

$$f(h) \leqslant 4^{\mathrm{wcol}_{3h-3}(\mathrm{cover}(P))} = 2^{\mathcal{O}(h^3)}$$

height and dimension

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$$f(h) \leqslant 4^{\mathrm{wcol}_{3h-3}(\mathrm{cover}(P))} = 2^{\mathcal{O}(h^3)}$$

$$= \mathcal{O}(h^6)$$

(Kozik, PM, Trotter 2022)

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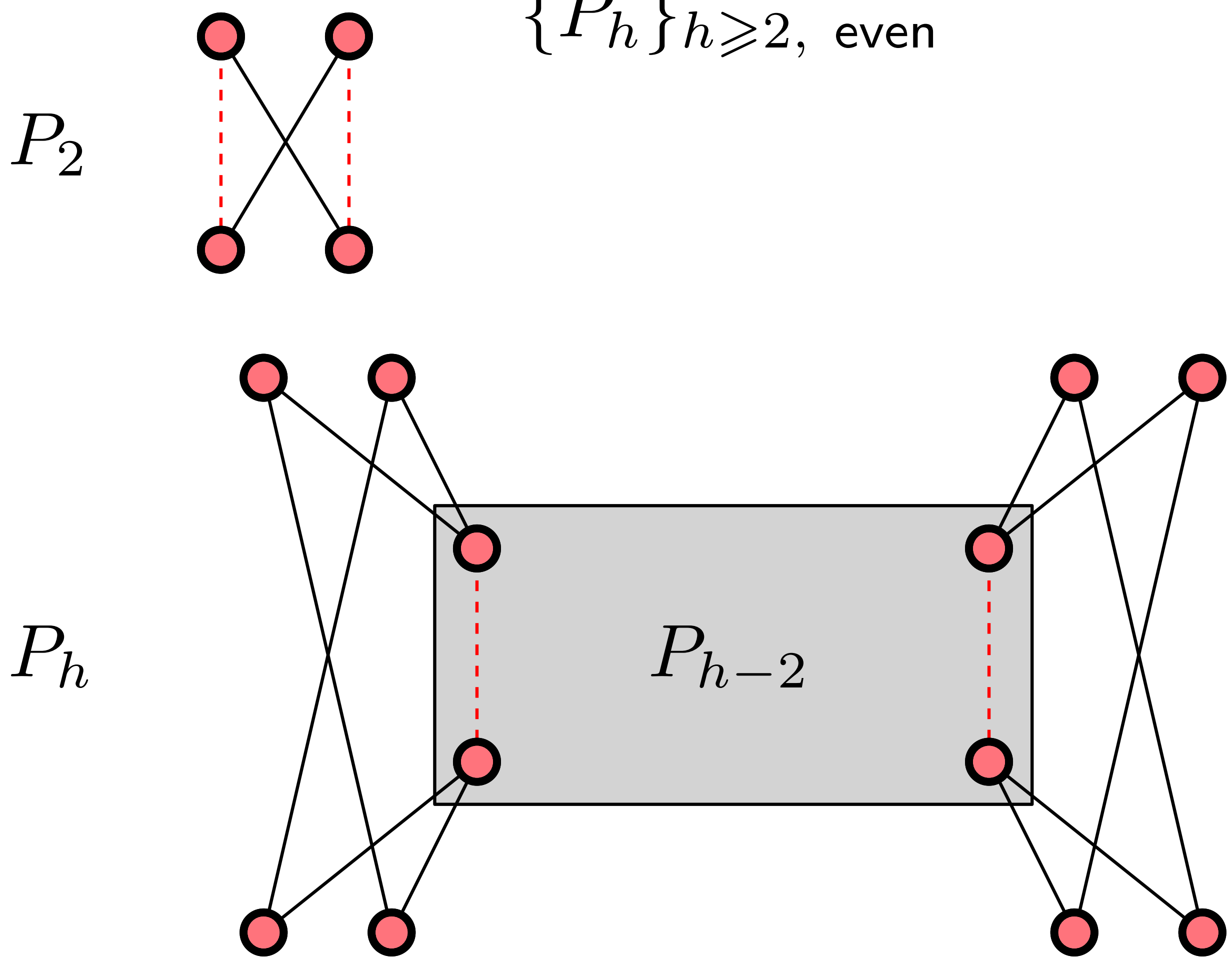
(Joret, PM, Ossona de Mendez, Wiechert 2019)

$$\begin{aligned} f(h) &\leqslant 4^{\mathrm{wcol}_{3h-3}(\mathrm{cover}(P))} = 2^{\mathcal{O}(h^3)} \\ &= \mathcal{O}(h^6) \quad \text{(Kozik, PM, Trotter 2022)} \\ &= \mathcal{O}(h^3) \quad \text{(Gorsky, Seweryn 2022)} \end{aligned}$$

Construction (Joret, PM, Wiechert 2017)

$\{P_h\}_{h \geqslant 2,\text{ even}}$



Construction (Joret, PM, Wiechert 2017)

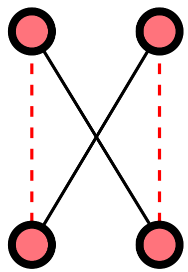
$\{P_h\}_{h \geqslant 2,\ \text{even}}$

$\text{height}(P_h) = h$

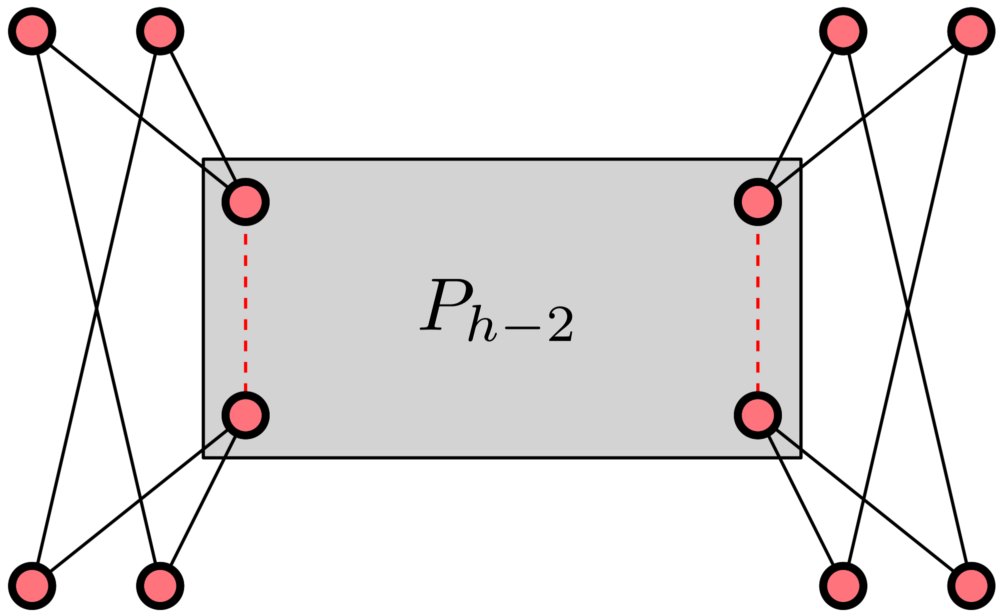
$\text{se}(P_h) \geqslant 2 \cdot \text{se}(P_{h-2})$

so $\dim(P_h) \geqslant 2^{h/2}$

P_2

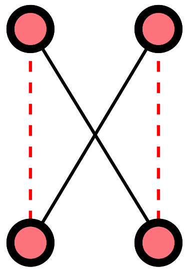


P_h



Construction (Joret, PM, Wiechert 2017)

P_2



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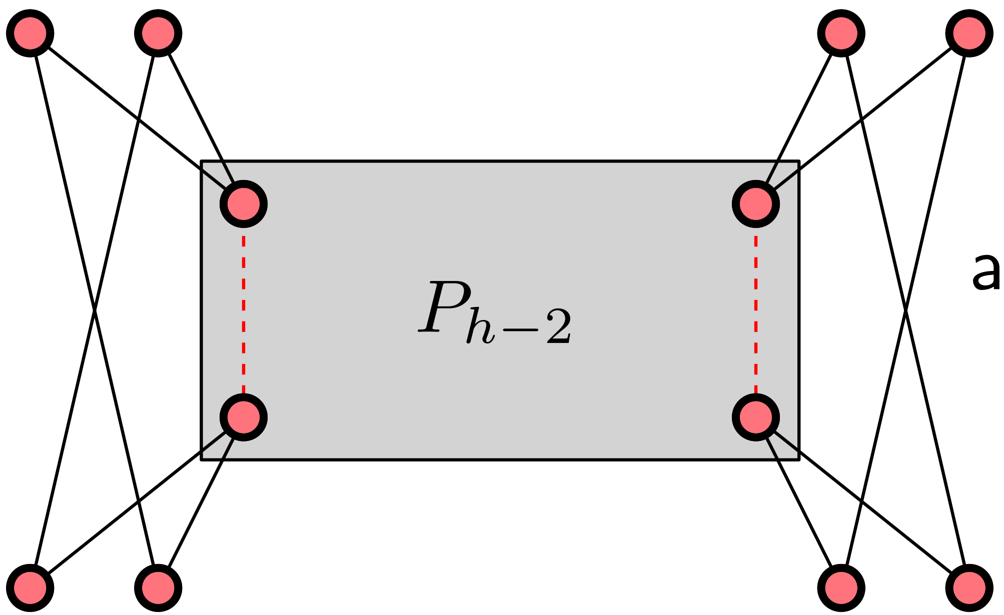
$\text{height}(P_h) = h$

$\text{se}(P_h) \geqslant 2 \cdot \text{se}(P_{h-2})$

so $\dim(P_h) \geqslant 2^{h/2}$

P_h



and $K_5 \not\preccurlyeq \text{cover}(P_h)$

$\text{tw}(\text{cover}(P_h)) \leqslant 3$

height and dimension

Theorem (Streib, Trotter 2014)
There is a function $f : \mathbb{N} \to \mathbb{N}$ such that for every $h \geqslant 1$ and every poset P of height $\leqslant h$ and with a planar cover graph, we have

$$\dim(P) \leqslant f(h)$$

Large dimensional posets are wide

Large dimensional planar posets are tall

(Joret, PM, Ossona de Mendez, Wiechert 2019)

$$\begin{aligned} f(h) &\leqslant 4^{\mathrm{wcol}_{3h-3}(\mathrm{cover}(P))} = 2^{\mathcal{O}(h^3)} \\ &= \mathcal{O}(h^6) \quad \text{(Kozik, PM, Trotter 2022)} \\ &= \mathcal{O}(h^3) \quad \text{(Gorsky, Seweryn 2022)} \end{aligned}$$

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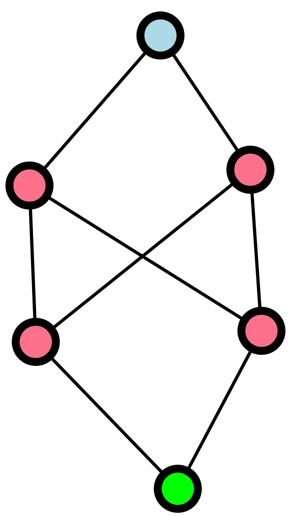
$= \mathcal{O}(h^6)$ (Kozik, PM, Trotter 2022)

$= \mathcal{O}(h^3)$ (Gorsky, Seweryn 2022)

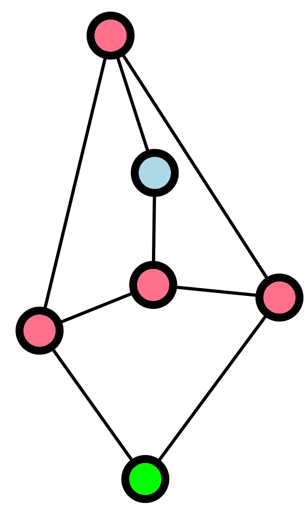
Theorem (Joret, PM, Wiechert 2017)
Let P be a poset with a planar diagram and height h.
Then

$$\dim(P) \leqslant 192h + 96$$

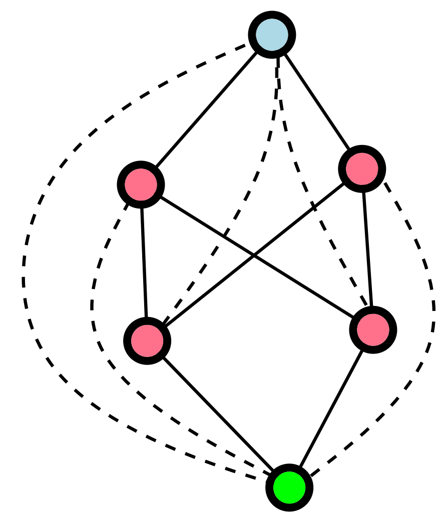
drawing posets



diagram

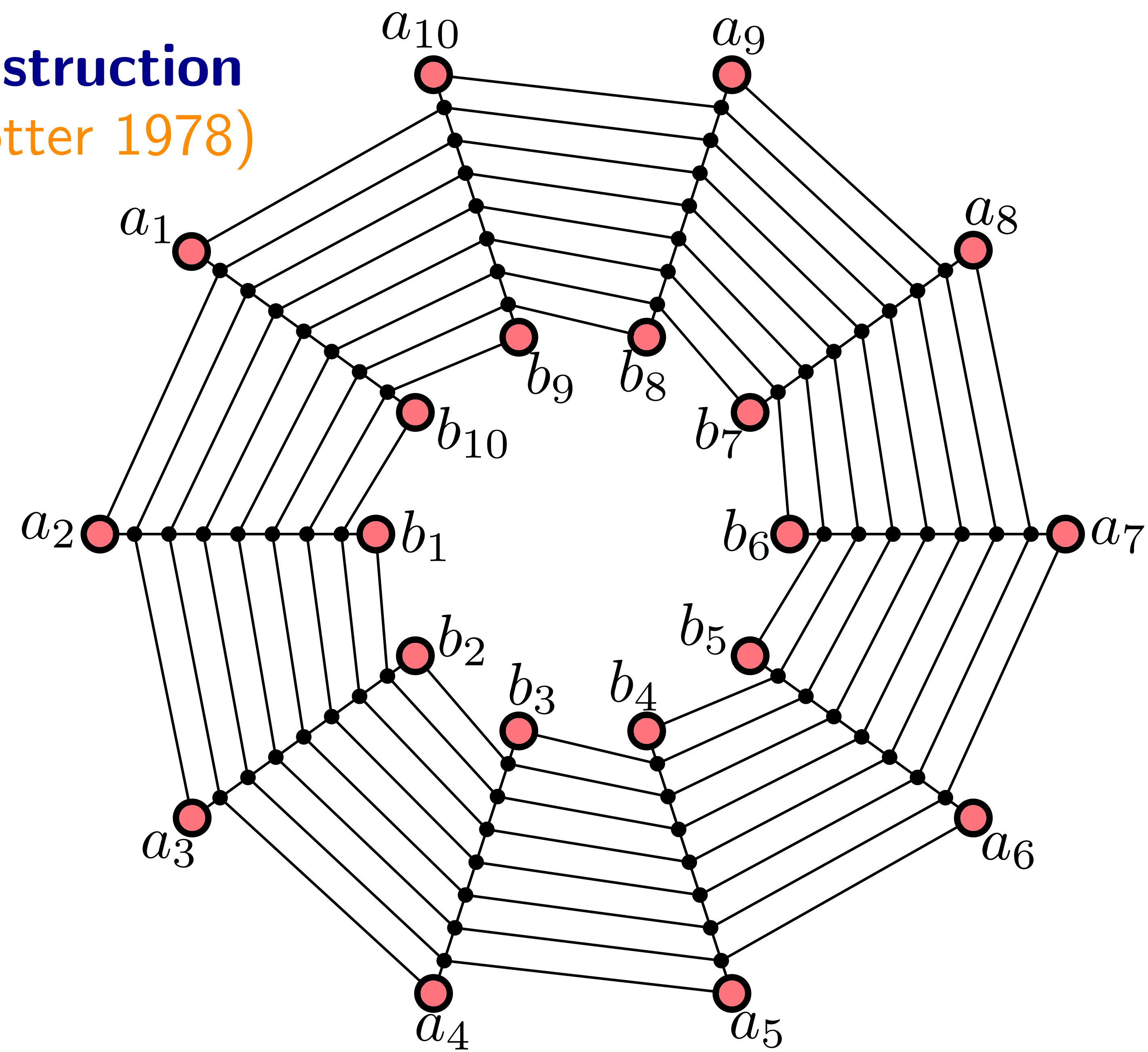


cover graph



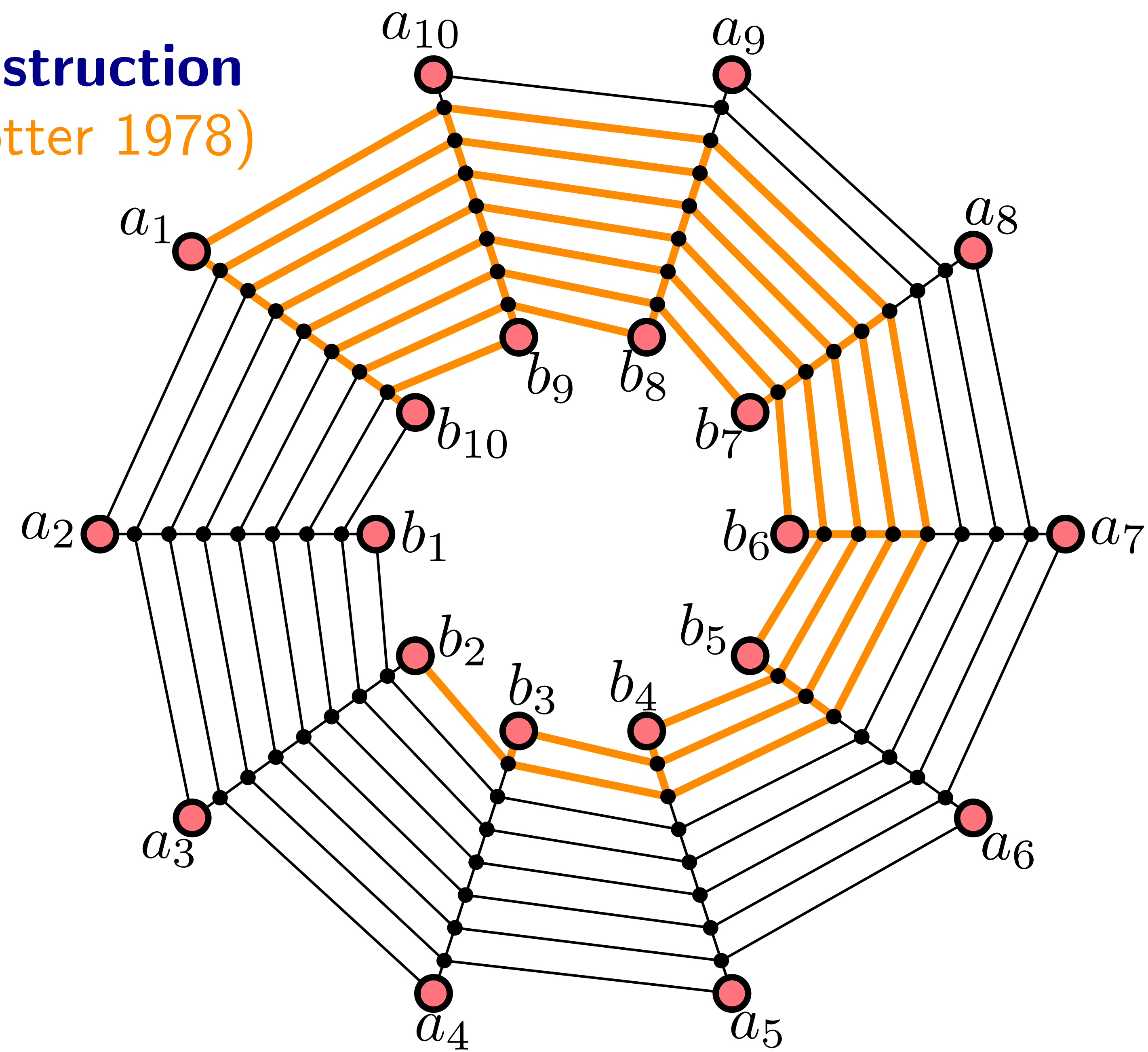
comparability graph

Construction
(Trotter 1978)

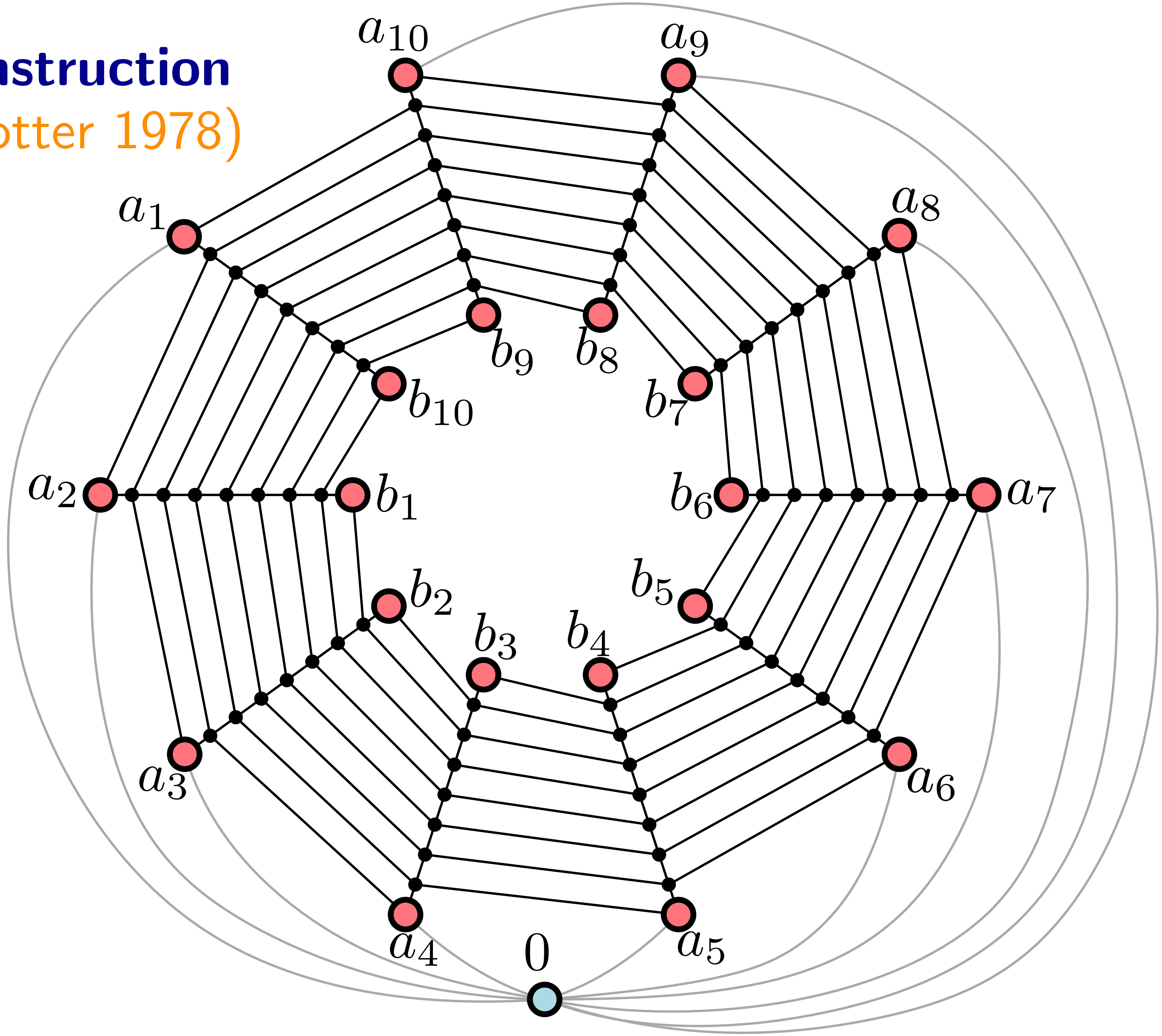


Construction

(Trotter 1978)



Construction
(Trotter 1978)

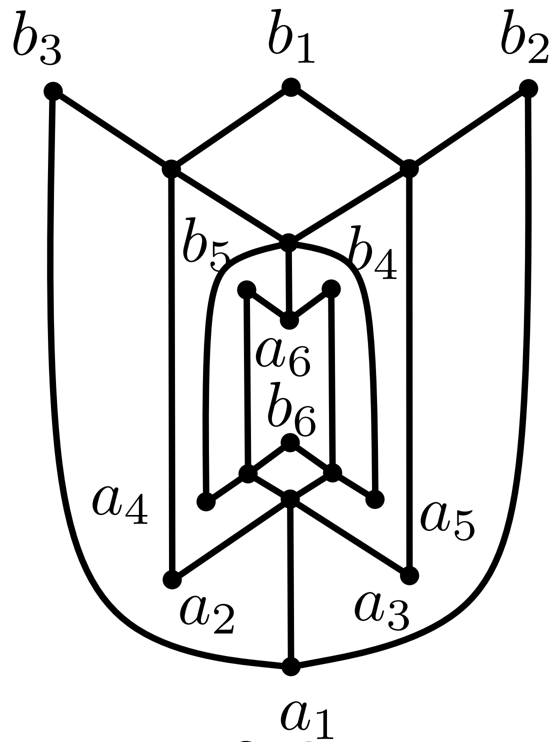


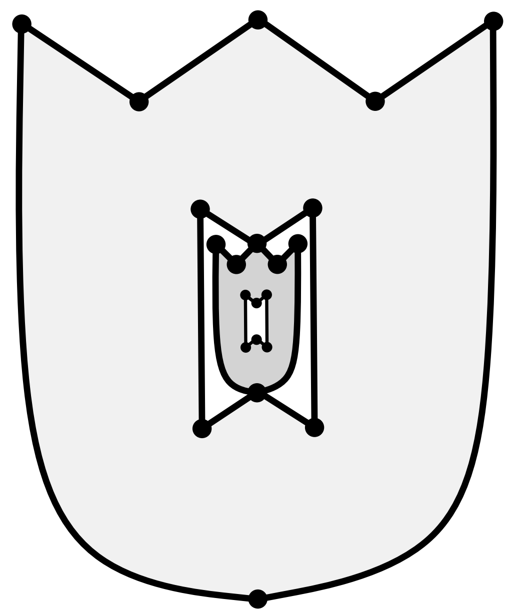
height and dimension

Construction (Joret, PM, Wiechert 2017)

For every $h \geqslant 1$, there is a poset P with a planar diagram, of height h and

$$\frac{4}{3}h - 2 \leqslant \dim(P)$$





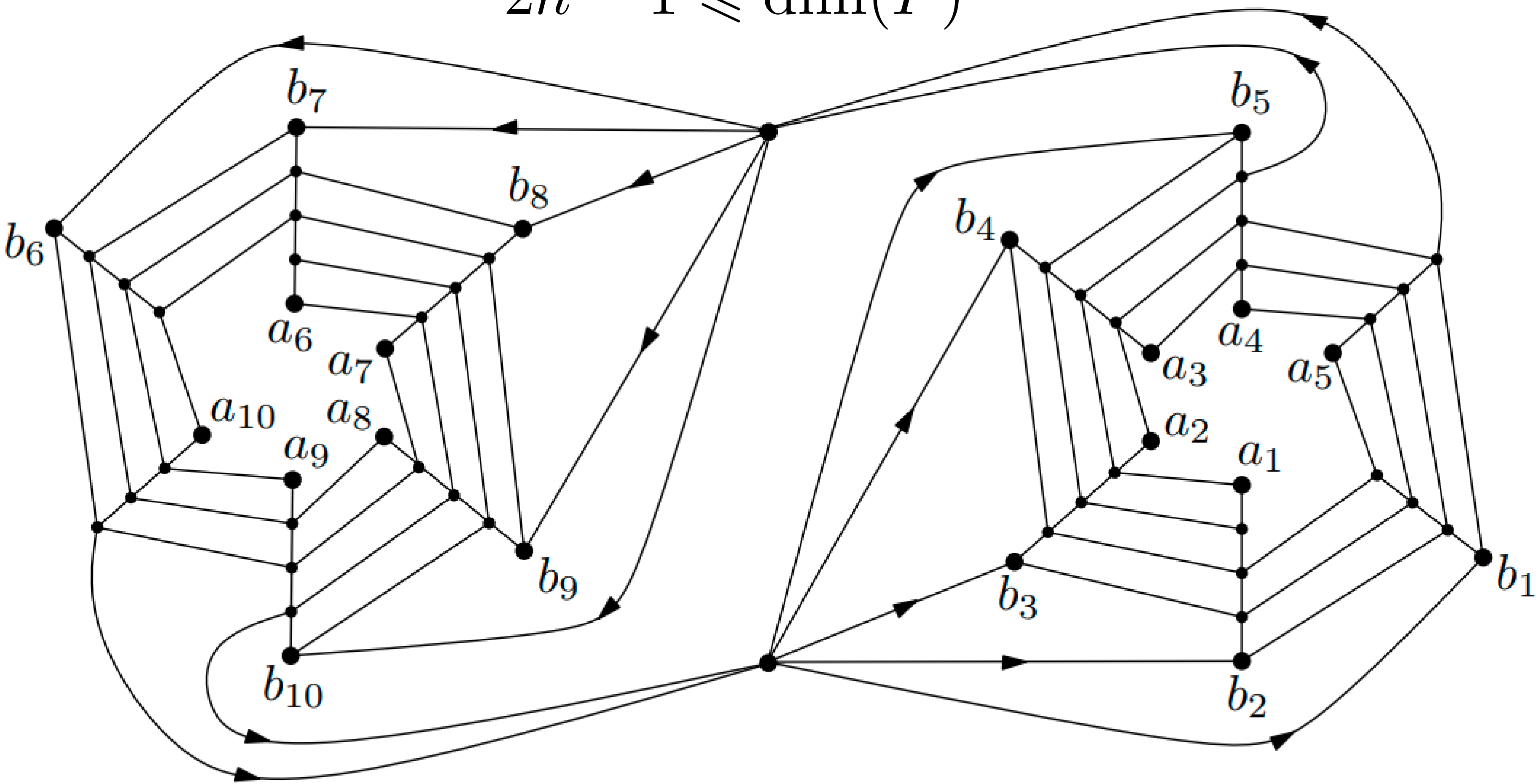
In each iteration of the construction we increase the height by 3 and the size of a standard example by 4

height and dimension

Construction (Joret, PM, Wiechert 2017)

For every $h \geqslant 1$, there is a poset P with a planar cover graph, of height h and

$$2h - 1 \leqslant \dim(P)$$



height and dimension

What is the maximum dimension of a poset with height h and

planar cover graph?	$2h-1$	$\mathcal{O}(h^3)$
planar diagram?	$\frac{4}{3}h-2$	192h+96

height and dimension

Theorem (Joret, PM, Ossona de Mendez, Wiechert 2019)
Let P be a poset of height at most h with a cover graph G such that $\mathrm{wcol}_{3h-3}(G) \leqslant c$. Then

$$\dim(P) \leqslant 4^c$$

$\mathrm{cover}(P)$	$\dim(P)$
genus $\leqslant g$	$= 2^{\mathcal{O}(h^3)}$
treewidth $\leqslant t$	$= 2^{\mathcal{O}(h^t)}$
	$= 2^{\Omega(h^{(t-1)/2})}$
no K_t minor	$= 2^{\mathcal{O}(h^{t-1})}$
no K_t top. minor	$= 2^{2^{\mathcal{O}(h \log h)}}$

height and dimension

wcol_r are capturing a widely studied measures of sparsity

$$\exists_f \forall_{\substack{G \in \mathcal{C} \\ r \geqslant 0}} \quad \mathrm{wcol}_r(G) \leqslant f(r) \quad \Leftrightarrow$$ $\mathcal{C}$ has bounded expansion

$$\forall_{\varepsilon > 0} \forall_{\substack{G \in \mathcal{C} \\ r \geqslant 0}} \quad \mathrm{wcol}_r(G) = \mathcal{O}(n^{\varepsilon}) \Leftrightarrow$$ $\mathcal{C}$ is nowhere dense

Theorem (Joret, PM, Ossona de Mendez, Wiechert 2019)
Let P be a poset of height at most h with a cover graph G such that $\mathrm{wcol}_{3h-3}(G) \leqslant c$. Then

$$\dim(P) \leqslant 4^c$$

Corollary $\forall_{\mathcal{C} \text{ class with bounded expansion}} \forall_{h \geqslant 1}$

posets of height h whose cover graphs are in $\mathcal{C}$ have bounded dimension

height and dimension

Theorem (Joret, PM, Ossona de Mendez, Wiechert 2019)
Let $\mathcal{C}$ be a monotone class of graphs. Then

$\mathcal{C}$ is nowhere dense $\iff$ $\forall_{h \geqslant 1} \exists_{\varepsilon > 0}$
n-element posets of height $\leqslant h$
whose cover graphs are in $\mathcal{C}$
have dimension in $\mathcal{O}(n^{\varepsilon})$

height and dimension

Theorem (Joret, PM, Ossona de Mendez, Wiechert 2019)
Let $\mathcal{C}$ be a monotone class of graphs. Then

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Conjecture
Let $\mathcal{C}$ be a monotone class of graphs. Then

$\mathcal{C}$ has bounded expansion $\iff$ $\forall_{h \geqslant 1}$
posets of height $\leqslant h$
whose cover graphs are in $\mathcal{C}$
have bounded dimension

height and dimension

Conjecture

Let $\mathcal{C}$ be a monotone class of bipartite graphs
that have bounded dimension.
Then graphs in $\mathcal{C}$ have bounded average degree.

height and dimension

Conjecture
Let $\mathcal{C}$ be a monotone class of bipartite graphs
that have bounded ~~dimension~~. boxicity
Then graphs in $\mathcal{C}$ have bounded average degree.

height and dimension

Conjecture

Let $\mathcal{C}$ be a monotone class of bipartite graphs
that have bounded ~~dimension~~. boxicity
Then graphs in $\mathcal{C}$ have bounded average degree.

Problem

$\forall_{d \geqslant 1} \; \exists_{f(d)}$ (bipartite)
every graph with minimum degree $\geqslant f(d)$
has a subgraph with boxicity $\geqslant d$

$\dim$-boundedness

Recall When $I \subseteq \mathrm{Inc}(P)$

I is reversible $\iff$ I has no alternating cylce

$\mathcal{H}(P)$ hypergraph: vertices $\equiv$ $\mathrm{Inc}(P)$

edges $\equiv$ alternating cycles

$$\dim(P) = \chi(\mathcal{H}(P))$$

cliques $\equiv$ standard examples

$\dim$-**boundedness**

$\mathcal{C}$ class of graphs

$\mathcal{C}$ is χ-bounded if there is a function f

such that for every $G \in \mathcal{C}$

$\chi(G) \leqslant f(\omega(G))$

$\dim$-boundedness

$\mathcal{C}$ class of graphs

$\mathcal{C}$ is χ-bounded if there is a function f

such that for every $G \in \mathcal{C}$

$\chi(G) \leqslant f(\omega(G))$

$\mathcal{C}$ class of posets

$\mathcal{C}$ is $\dim$-bounded if there is a function f

such that for every P with a cover graph in $\mathcal{C}$

$\dim(P) \leqslant f(\mathrm{se}(P))$

$\dim$-boundedness

Conjecture Posets with planar cover graph are $\dim$-bounded

$\dim$-boundedness

Conjecture Posets with planar cover graph are $\dim$-bounded

first published reference: Trotter (1992)
informal comment on p. 119

$\dim$-boundedness

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Theorem (Joret, PM, Walczak 2018; never written)
Posets with cover graphs of bounded pathwidth
are $\dim$-bounded

$\dim$-**boundedness**

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Theorem (Joret, PM, Walczak 2018; never written)
Posets with cover graphs of bounded pathwidth
are $\dim$-bounded

Theorem (Joret, PM, Mi. Pilipczuk, Walczak 2022+)
Posets with cover graphs of bounded treewidth
are $\dim$-bounded

$\dim$-boundedness

Conjecture Posets with planar cover graph are $\dim$-bounded
first published reference: Trotter (1992)
informal comment on p. 119

Theorem (Joret, PM, Walczak 2018; never written)
Posets with cover graphs of bounded pathwidth
are $\dim$-bounded

Theorem (Joret, PM, Mi. Pilipczuk, Walczak 2022+)
Posets with cover graphs of bounded treewidth
are $\dim$-bounded

Theorem (PM, Smith-Blake, Trotter 2022+)
Posets with planar cover graph and a unique minimal element
are $\dim$-bounded

In particular, for every such poset P: $\dim(P) \leqslant 2\,\mathrm{se}(P) + 2$

$\dim$ vs se continued

Theorem (Biró, Hamburger, Pór 2015)
For every fixed $t \geqslant 3$, the maximum dimension of an n-element S_t-free poset is $o(n)$

Theorem (Biró, Hamburger, Kierstead, Pór, Trotter, Wang 2020)
For every fixed $t \geqslant 3$ and every large enough n, there is an n-element S_t-free poset Q with

$$\dim(Q) \geqslant \frac{n^{1-\frac{2t-1}{t(t-1)}}}{\log n}$$