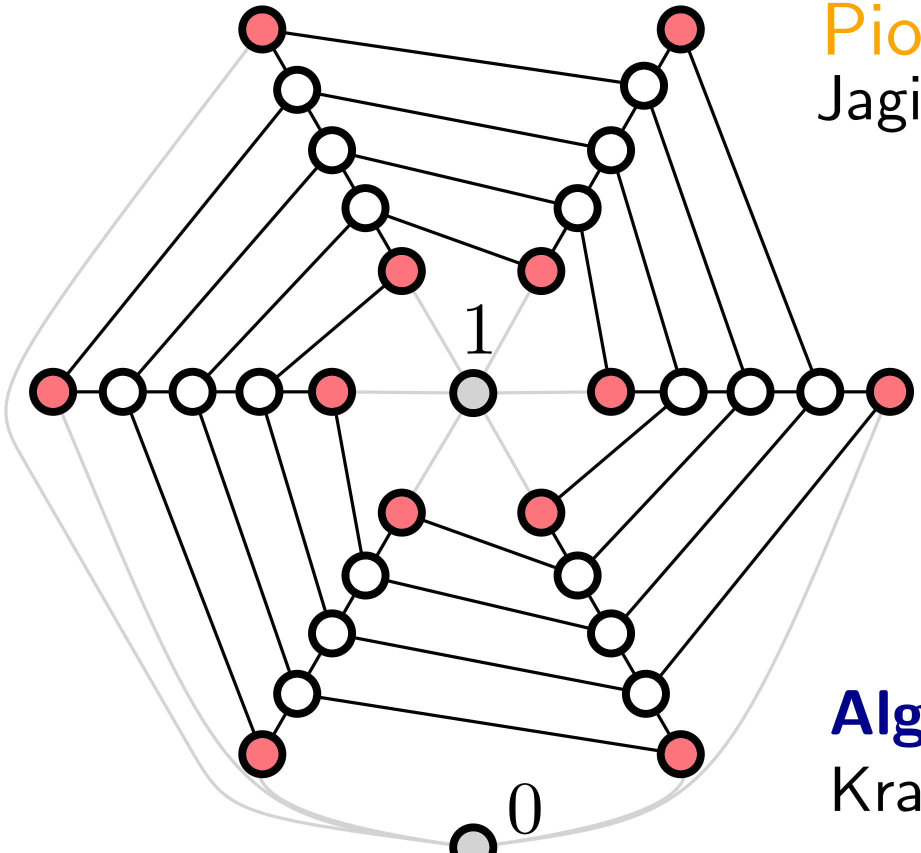
Combinatorics of posets

Lecture 4: max. asymptotics of dimension vs fixed se



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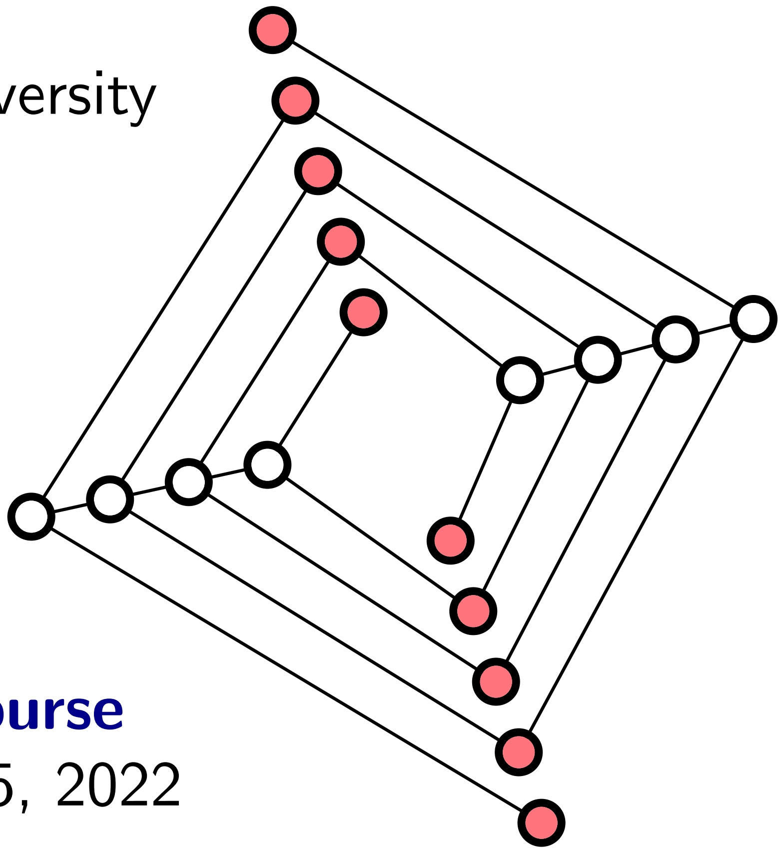
Jagiellonian University

AlgoMaNet course

Kraków, May 25, 2022



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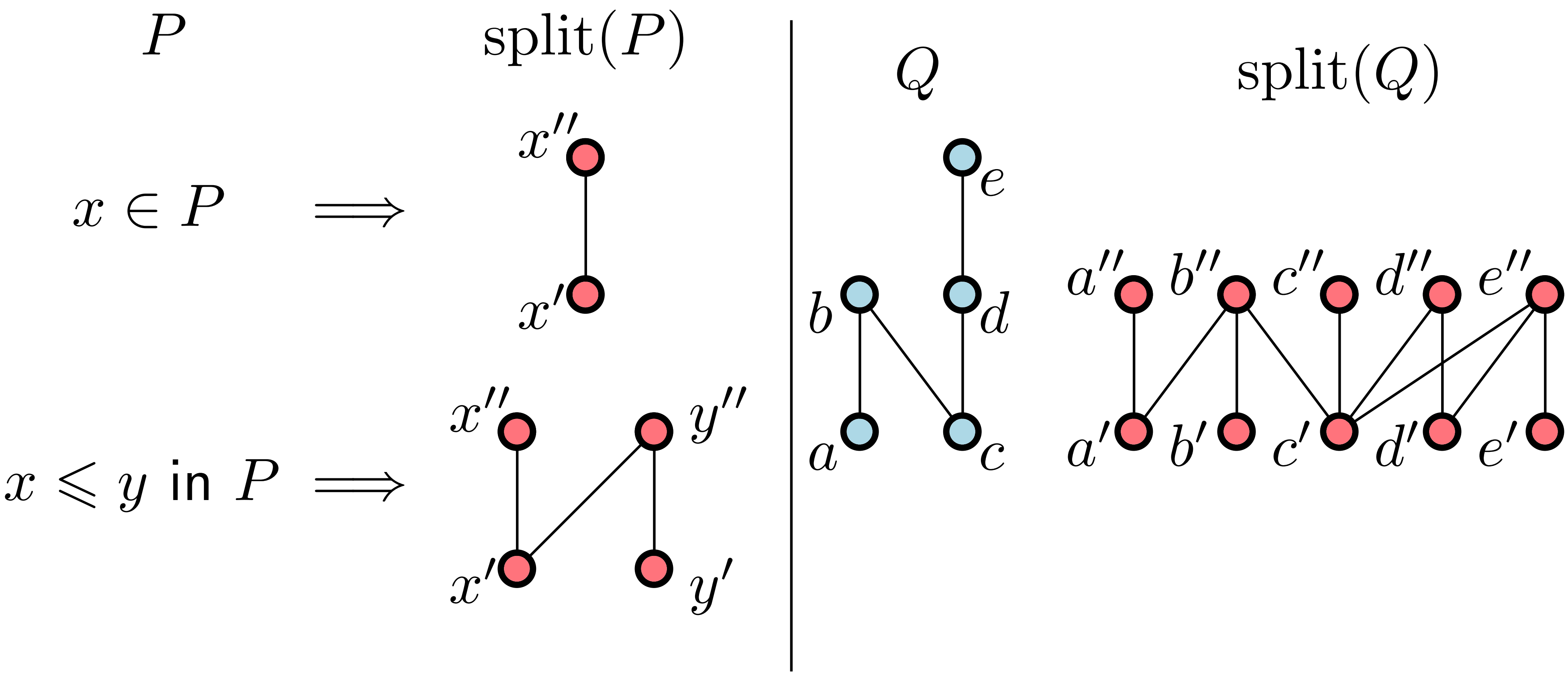
For every fixed $t \geqslant 3$ and every large enough n, there is an n-element S_t-free poset Q with

$$\dim(Q) \geqslant \frac{n^{1-\frac{2t-1}{t(t-1)}}}{\log n}$$



Exercise $\dim(P) \leqslant \dim(\mathrm{split}(P)) \leqslant \dim(P) + 1$

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otw $\exists_{a \in P}$ such that a' and a'' in the copy of S_t in Q

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contradiction

$b' < a''$ in Q
$a' < c''$ in Q $\implies$ $b < a < c$ in P $\implies$ $b' < c''$ in Q

Lemma Let t, q be positive integers with $2 \leqslant q$, $t \leqslant q$, and let $r \geqslant t2^t \ln q$. Let $X = [x_{ij}]$ be a random $r \times q$ binary matrix, in which each entry is 1 with probability $\frac{1}{2}$.
Let E be the event that

$$\forall_{1 \leqslant j_1 < j_2 < \cdots < j_t \leqslant q} \; \exists_{i \in [r]} \text{ s.t. } \forall_{\ell \in [t]} \; x_{j_\ell} = 1 \text{ and } \forall_{\substack{k \in [t] \\ k \neq \ell}} x_{j_k} = 0$$

Then $\mathrm{Prob}(E) > 0$.

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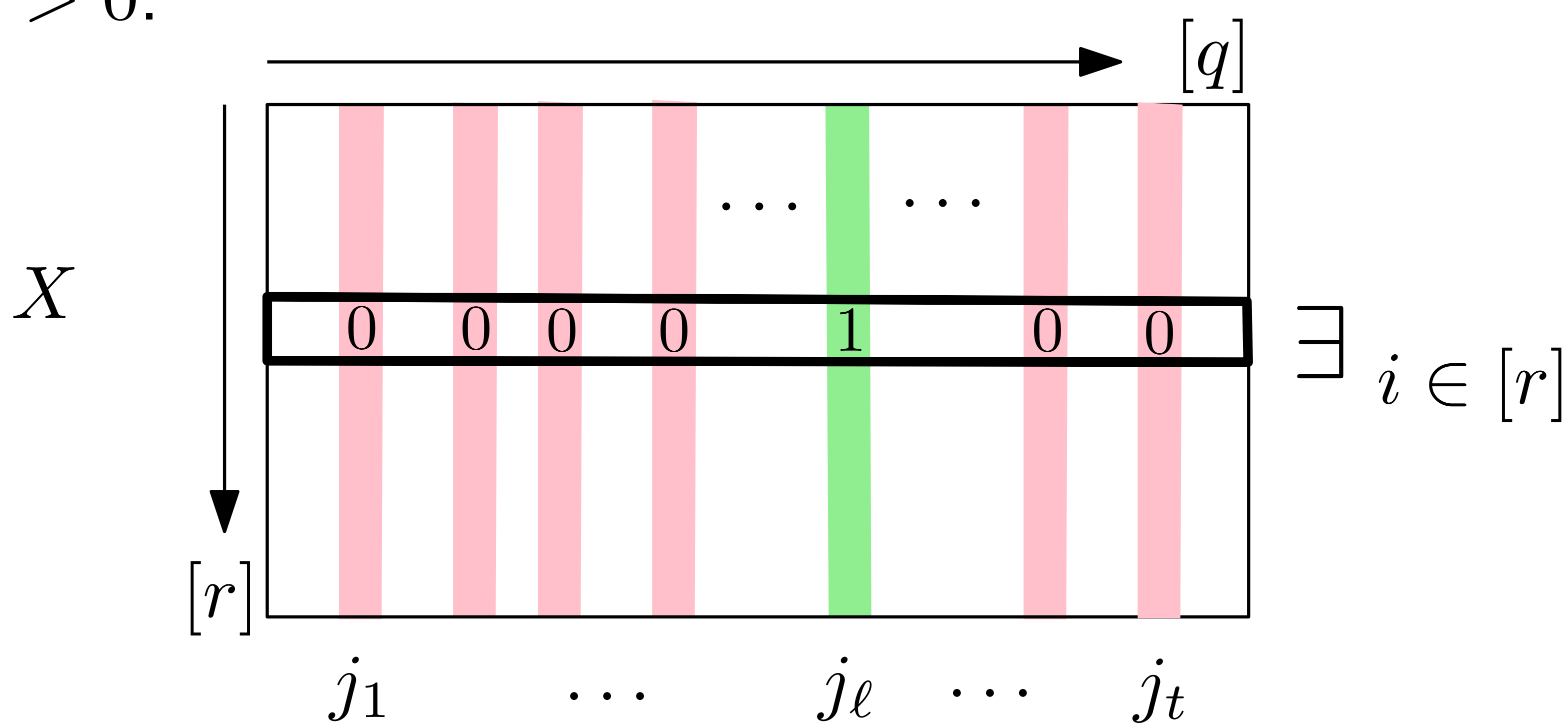

Proof Fix $s = (j_1, \ldots, j_t)$ and $\ell \in [t]$

$E_{s,\ell}$ the event that at least one row has the property wrt. s and ℓ

$$\mathrm{Prob}(\overline{E_{s,\ell}}) = (1 - 2^{-t})^r$$

$$\begin{aligned}\mathrm{Prob}(E) &= \mathrm{Prob}(\bigcap E_{s,\ell}) = 1 - \mathrm{Prob}(\bigcup \overline{E_{s,\ell}}) \\ &\geqslant 1 - \sum \mathrm{Prob}(\overline{E_{s,\ell}}) \\ &\geqslant 1 - \binom{q}{t} \cdot t \cdot (1 - 2^{-t})^r > 1 - q^t \cdot e^{-2^{-t} r} \\ &= 1 - e^{t \ln q} \cdot e^{-2^{-t} r} \\ &\geqslant 1 - e^{t \ln q} \cdot e^{-2^{-t} \cdot t 2^t \ln q} \\ &\geqslant 0\end{aligned}$$

$1 - x \leqslant e^{-x}$

Theorem (Biró, Hamburger, Pór 2015)
For every fixed $t \geqslant 3$, the maximum dimension of an n-element S_t-free poset is $o(n)$

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Q the split of P
$2n$-element poset
height 2
S_t-free as well

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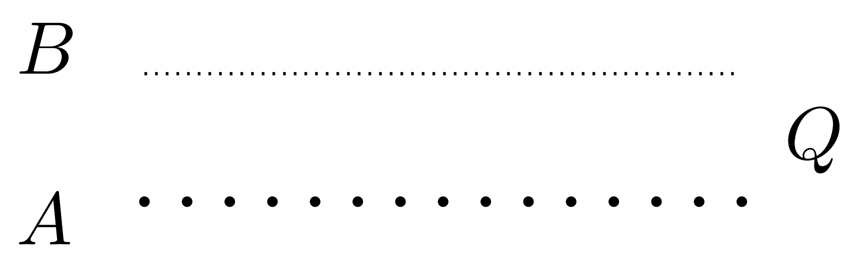

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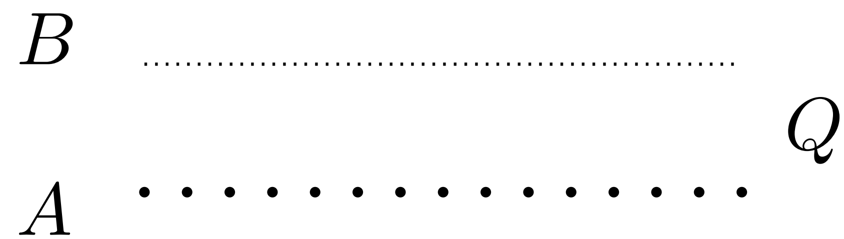

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consider $S = \{a_1, \ldots, a_t\} \subseteq A$
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$b \in B$ is an i-mate of S if

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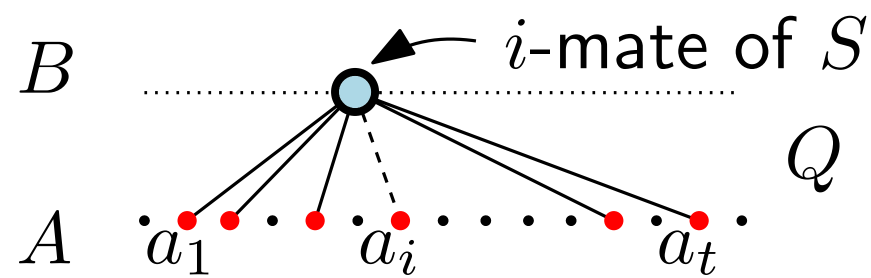

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let $|C| = q$ and $C = \{c_1, \ldots, c_q\}$
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given a row $x = (x_1, \ldots, x_q)$ of X construct two permutations of A

σ_{left}	$A - C$	$\{c_i \mid x_i = 0\}$	$\{c_i \mid x_i = 1\}$ sorted right-to-left
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σ_{left}, σ_{right} $\Longrightarrow$ L_{left}, L_{right} push elements of B as low as possible

Claim All incomparable pairs in $C \times B$ can be reversed with $q' = 2 \cdot \lceil t \cdot 2^t \cdot \ln q \rceil$ linear extensions

Proof of the Claim continued

let R be the constructed set of q' lin. ext. of Q

Goal:

$$\forall_{(c,b) \in C \times B} \text{ incomparable pair in } Q$$

$$\exists_{L \in R} \text{ such that } b < c \text{ in } L$$

(blackboard)

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Back to the main Proof $(\dim(Q) = o(n))$

inverse Ramsey numbers

# of colors

$$r(2, 2; n) \geqslant \tfrac{1}{2} \log n$$

size of subsets colored

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let $c(n) = r(t, t; g(n))$
$\lim c(n) = \infty$
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$$\dim(Q) \quad \leqslant \quad \tfrac{n}{c(n)} \cdot (2 \cdot \lceil t \cdot 2^t \cdot \log c(n) \rceil) + g(n) + 2$$

$$
\begin{aligned}
\dim(Q) &\leqslant \frac{n}{c(n)} \cdot (2 \cdot \lceil t \cdot 2^t \cdot \log c(n) \rceil) + g(n) + 2 \\
&\leqslant \alpha_t \cdot n \cdot \frac{\log c(n)}{c(n)} + g(n)
\end{aligned}
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Proof fix $t \geqslant 3$

random poset Q
$2n$ elements
height-2

B . . . . n elements

A . . . . n elements

$\forall$ $a \in A$, $b \in B$

$\mathrm{Prob}(a < b) = p$

$\mathrm{Prob}(a \parallel b) = 1 - p = q$

fix $p = \frac{1}{n^\alpha}$

$\alpha \in (0, 1)$ constant depending on t

definitely $p < 1/2$

Claim 1 Given $q \geqslant \frac{1}{2}$ $(q \geqslant \frac{\log^2 n}{n})$

$$\text{Prob}\left(\exists \begin{array}{c}\text{a perfect matching}\\ \text{between } A \text{ and } B \text{ in}\\ \text{the incomparability graph of } Q\end{array}\right) \to 1$$

Claim 2

$$\text{Prob}\left(\exists \begin{array}{c} A' \subseteq A \\ B' \subseteq B \end{array} \text{ such that } \begin{array}{c} |A'| \geqslant \frac{3 \log n}{p} \\ |B'| \geqslant \frac{3 \log n}{p} \end{array} \text{ and } A' \parallel B' \text{ in } Q\right) \to 0$$

Claim 3

$$E\left(\# \text{ of copies of } S_t \text{ in } Q\right) \leqslant \frac{n}{6}$$