PARTITE LEMMA OF NEŠETRIL AND RÖDL WITH NEW NOTATION

A matrix is called row-constant if for any row, all elements in that row are the same. A matrix is called column-constant of for any column, all elements in that column are the same. Let A be a finite set. Let $S \subseteq A^n$ and let M be a matrix formed by vectors from S as rows. We say that S is combinatorial line if it has $|A|$ distinct elements and such that there is a non-empty subset of columns of M so that M restricted to these columns is row-constant, and M restricted to the remaining columns is column-constant.

Example $A = \{1, 2, 3\}$.

$$\begin{pmatrix} \mathbf{1} & 3 & 1 & \mathbf{1} & 1 \\ \mathbf{2} & 3 & 1 & \mathbf{2} & 1 \\ \mathbf{3} & 3 & 1 & \mathbf{3} & 1 \end{pmatrix}$$

This matrix corresponds to a combinatorial line with elements $(1, 3, 1, 1, 1)$, $(2, 3, 1, 2, 1)$, $(3, 3, 1, 3, 1)$. Here the first and fourth columns give a row-constant matrix, and the rest give a column-constant matrix.

Theorem 1 (Hales-Jewett, [2])**.** *For any finite set A, and positive integer c, there is a positive integer r such that if A^r is colored with c colors, then there is a monochromatic combinatorial line.*

Proof. We shall assume that $A = [m]$ for a positive integer m. We say that a family of combinatorial lines is color-focused if they share the same last element (after each line is ordered lexicographically), and each line without the last element is monochromatic. Observe that if there is a color-focused family of c lines in a c-coloring of $[m]^n$, then there is a monochromatic line.

We shall prove a stronger induction statement.

Claim For each $r \leq c$, there is n such that whenever $[m]^n$ is c-colored, there is either a monochromatic line or a color-focused family of r lines. Denote such $n = HJ(m, c, r)$ and observe that if the claim holds for m, then one can guarantee a monochromatic line by taking $r = c$.

Use double induction on m, with $m = 1$ being trivial, and r. Assume that $m > 1$, assume that the claim holds for all smaller values of m and any $r \leq c$. Fix m, and assume that the statement holds for all smaller values of m. If $r = 1$, take $n = HJ(m-1, c, c)$. Then considering only the elements $[m-1]^n \subseteq [m]^n$, we see by induction that there is a monochromatic line in $[m-1]^n$, that is a line without the last element in $[m]^n$. If $r > 1$, $r \leq c$, let $n' = HJ(m, c, r-1)$ and $n'' = HJ(m-1, c^{m^{n'}}, c^{m^{n'}})$, take $n = n' + n''$. Consider a c-coloring of $[m]^n$. If there is a monochromatic line, we are done, otherwise assume that there is no such a line and we shall show the existence of a color-focused family with r lines. Identify $[m]^n$ with $[m]^{n'} \times [m]^{n''}$. There are $c^{m^{n'}}$ ways to color $[m]^{n'}$. Consider a coloring f'' of $[m]^{n''}$ with $c^{m^{n'}}$ colors, where the n''-tuple $\mathbf{a}$ gets the color-set of $[m]^{n'} \times \mathbf{a}$. Since $n'' = HJ(m-1, c^{m^{n'}}, c^{m^{n'}})$, there is a monochromatic line $L'' = (P''_1, P''_2, \dots, P''_{m-1})$ in $[m-1]^{n''} \subseteq [m]^{n''}$, it is equal to a combinatorial line in $[m]^{n''}$ without the last element P'_-. The color of L'' corresponds to a coloring f' of $[m]^{n'}$. Since $n' = HJ(m, c, r-1)$, there is a color-focused family $L'_1 = (P'_{1,1}, P'_{1,2}, \dots, P'_{1,m}), \dots, L'_{r-1} = (P'_{r-1,1}, \dots, P'_{r-1,m})$ in $[m]^{n'}$ of size $r-1$. Let $P'_- = P_{i,m}$, $i = 1, \dots, r-1$ be the minimal element of all these lines.

Now, the following is a color-focused family of r lines in $[m]^n$:

$$L_1 = (P'_{1,1}P''_1,\ P'_{1,2}P''_2,\ \dots,\ P'_{1,m-1}P''_{m-1},\ P'_-P''_-),$$
$$L_2 = (P'_{2,1}P''_1,\ P'_{2,2}P''_2,\ \dots,\ P'_{2,m-1}P''_{m-1},\ P'_-P''_-),$$
$$\cdots$$
$$L_{r-1} = (P'_{r-1,1}P''_1,\ P'_{r-1,2}P''_2,\ \dots,\ P'_{r-1,m-1}P''_{m-1},\ P'_-P''_-),$$
$$L_r = (P'_-P''_1, P'_-P''_2, \dots, P'_-P''_{m-1}, P'_-P''_-).$$

□

Note, that the smallest such r for Hales-Jewett theorem in case when $|A| = 3$ and $c = 2$ is equal to 4, see [3].

Theorem 2 (A density version of Hales-Jewett theorem by Furstenberg-Katznelson [1])**.** *For any finite set A and a positive real number ϵ, there is $N = N(\epsilon, |A|)$ such that for any $n > N$ and any subset S of A^n such that $|S| > \epsilon |A^n|$, S contains a combinatorial line.*

A hypergraph is k-partite if its vertex set can be partitioned in k-parts such that each hyperedge contains at most one vertex from each part. We call the parts vertex-parts. A k-uniform hypergraph is called a k-graph. We interpret k-partite k-graphs as a set of k-tuples. By allowing independent vertices, we can assume that all parts of a k-partite hypergraph are of the same size. If we associate each part with a set X, then each hyperedge corresponds to a k-tuple over an alphabet X. Thus, we can talk about sets of tuples, or vector systems instead of partite hypergraphs. This allows to use a much less cluttered notation. We say that two sets of k-tuples over an alphabet X are isomorphic, if the corresponding hypergraphs are isomorphic. Similarly, the subsystem of k-tuples and induced subsystem of k-tuples is defined. Recall, that a hypergraph H is an induced subhypergraph of H' if the vertex set V of H is a subset of vertex set of H' and the only edges induced by V in H' are the edges of H.

Theorem 3 (Nešetřil-Rödl, [4])**.** *For any k-partite k-graph H and any c there is a k-partite k-graph H' such that any c-coloring of edges of H' contains a monochromatic copy of H as an induced sub-graph.*

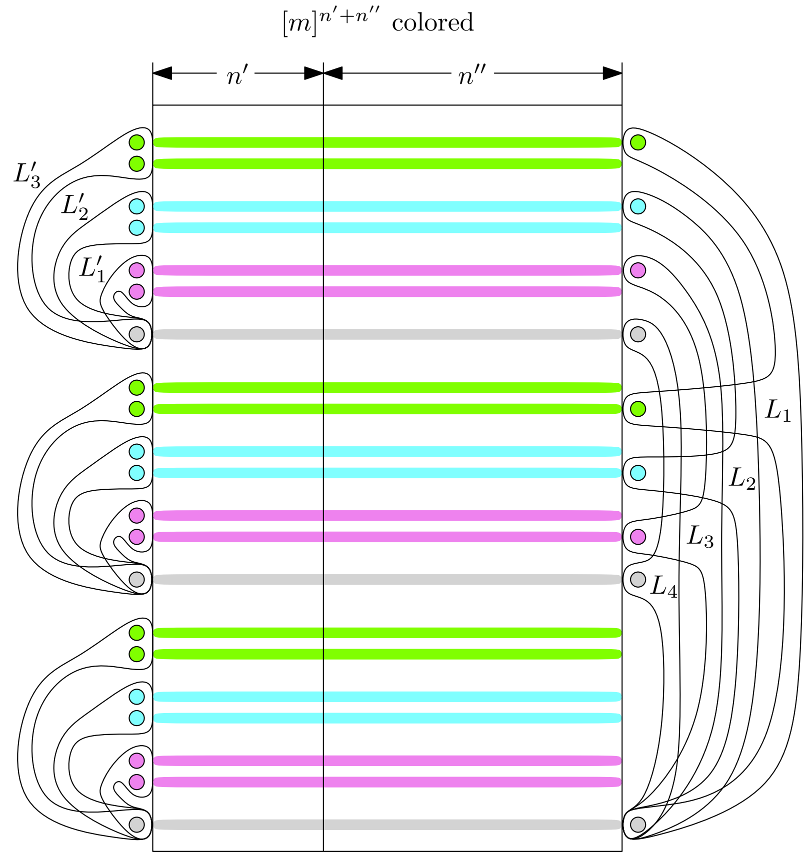

FIGURE 1. Example illustrating the proof of Hales-Jewett theorem with $r = 4$.

Proof. Instead of a hypergraph, we consider an equivalent set H of p k-tuples over alphabet X. Fix a constant r, specified later. We construct a system H' of k-tuples over alphabet $X' = X^r$, i.e., each letter of a new alphabet is an r-tuple of letters from X. Represent each such a k-tuple T as a $k \times r$ matrix $M(T)$ with elements from X, such that ith row is an ith component of the tuple. Let H' consist of all such k-tuples T so that each each column of $M(T)$ is a tuple from H. Note that each column in $M(T)$ is one of the possible p vectors from H. Thus, H' is equivalent to H^r. We claim that when r is sufficiently large, then H' satisfies the conditions of the theorem.

Consider a c coloring of H'. This coloring gives a coloring of H^r. If r is at least as large as the Hales-Jewett theorem requires, there is a monochromatic combinatorial line S. We claim that S corresponds to an induced copy of H in H'. Indeed, since S is a combinatorial line, it has p vectors, so that, without loss of generality, the matrix with these p vectors being the rows, is row-constant in the first two columns and column-constant in the remaining columns. The corresponding p matrices have distinct 1st and 2nd columns corresponding to distinct edges of H and for each $2 < i \leq r$, the i^{th} columns of these matrices are the same. Returning to rows of these matrices, that correspond to X', we see that the jth row of one matrix differs from the jth row of another matrix only in the first two coordinates. Restricted to the first coordinate, they give exactly a copy of H. Since there are no other elements of H except for these p ones, this copy of H is induced in H'. $\square$

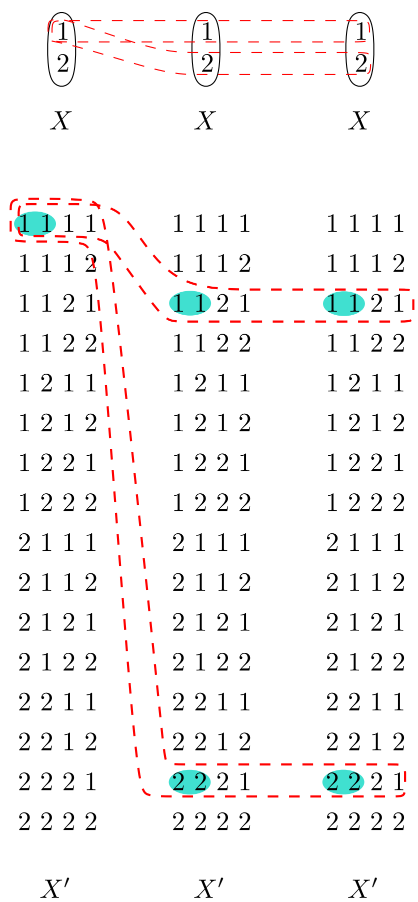

FIGURE 2. Systems H and H' from Example.

Let $k = 3, p = 2$, $X = \{1,2\}$, $H = \{(1,1,1),(1,2,2)\} = \{a_1, a_2\}$. Let $r = 4$. Then H' is a set of 4-tuples over an alphabet $X' = X^4 = \{1,2\}^4$. So, H' corresponds to a set of 3×4 matrices with each column being an element of H, and rows being elements of X^r. For example,

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 1 \\ 1 & 2 & 2 & 1 \end{pmatrix} = (a_1, a_2, a_2, a_1).$$

Thus H' corresponds to H^4. Coloring H^4 creates a monochromatic combinatorial line S, say (a_1, a_1, a_2, a_1), (a_2, a_2, a_2, a_1). Here, we see that in $\begin{pmatrix} a_1 & a_1 & a_2 & a_1 \\ a_2 & a_2 & a_2 & a_1 \end{pmatrix}$ the first two columns give a row-constant matrix and last two columns give a column-constant matrix. Returning to the corresponding tuples of H' written in a matrix form we see that S corresponds to

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 1 \\ 2 & 2 & 2 & 1 \end{pmatrix}.$$

We see that the matrices are identical restricted to the last two columns. The first two columns are the same in each of the matrices, and equal to the corresponding tuple of H.

References

[1] Furstenberg, H. and Katznelson, Y. *A density version of the Hales-Jewett theorem.* J. Anal. Math. 57 (1991), 64–119.

[2] Hales, A. W. and Jewett, R. I. *Regularity and positional games.* Trans. Amer. Math. Soc. 106 1963 222–229.

[3] Hindman N. and Tressler E. *The First Nontrivial Hales-Jewett Number is Four*, Ars Combinatoria 113 (2014), 385–390.

[4] Nešetřil J. and Rödl, V. *Two proofs of the Ramsey property of the class of finite hypergraphs.* European J. Combin. 3 (1982), no. 4, 347352.