Flexible Contact Representations

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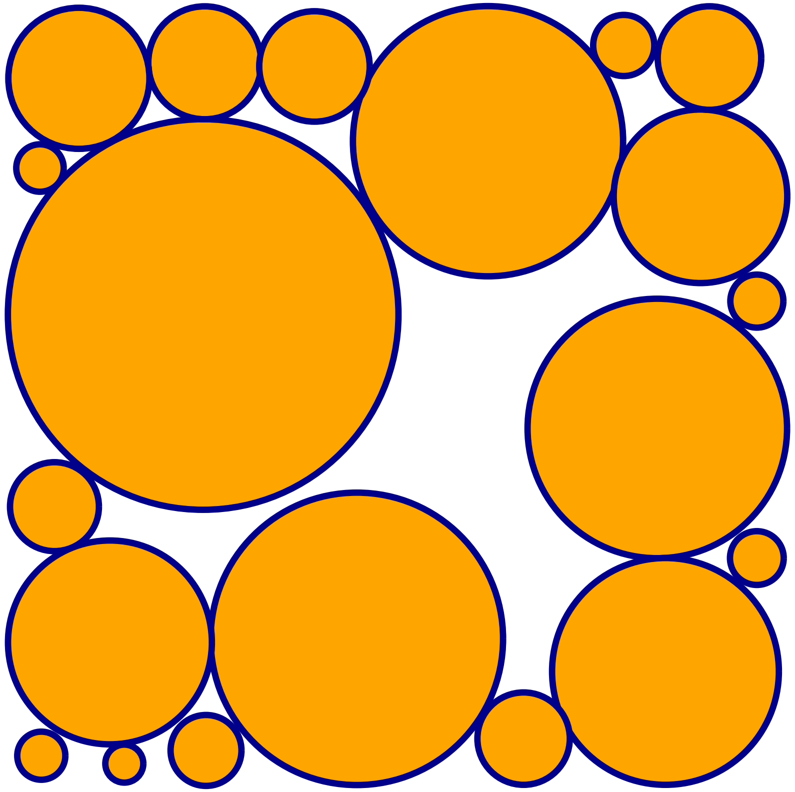

Contact Graphs

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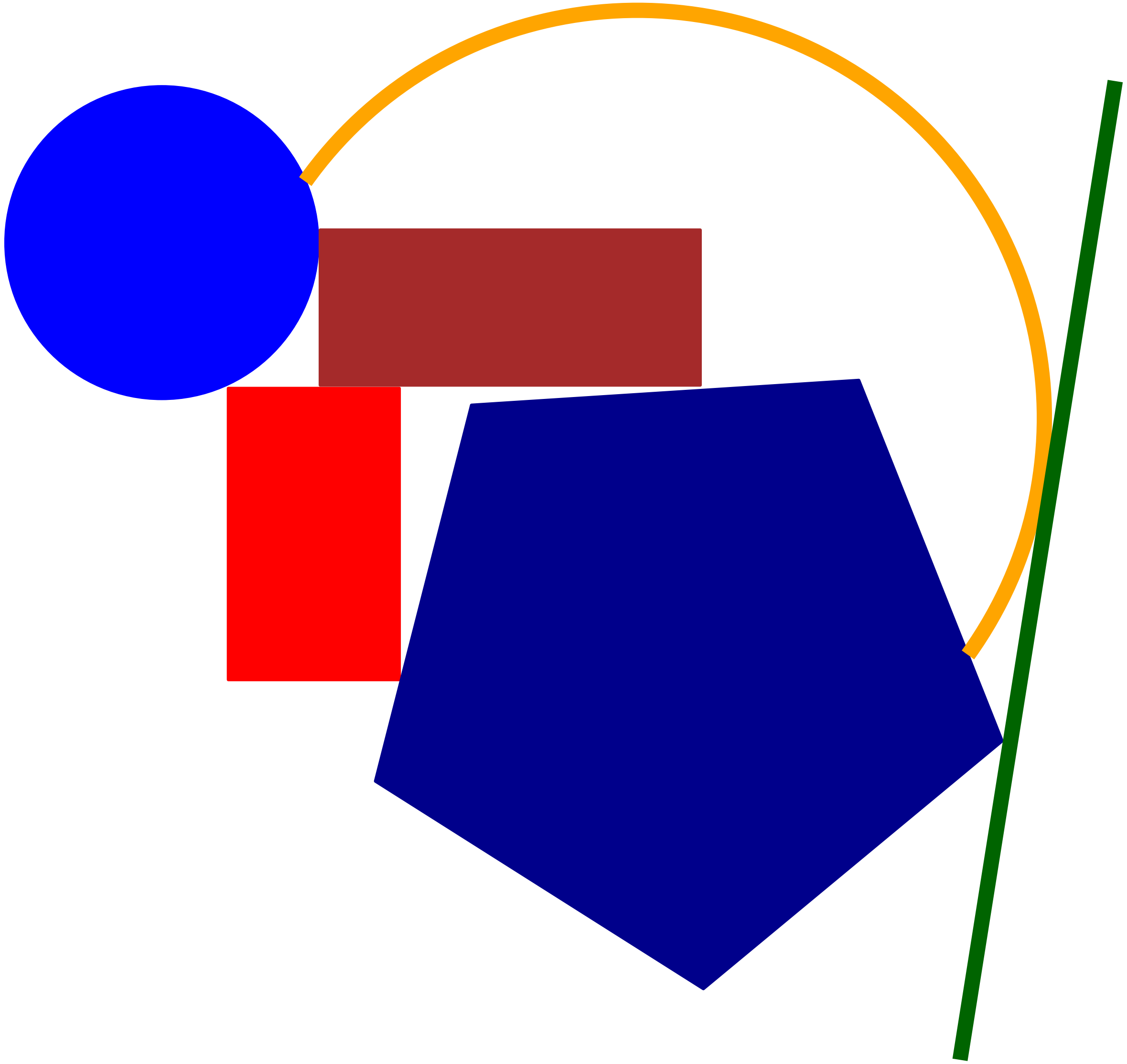

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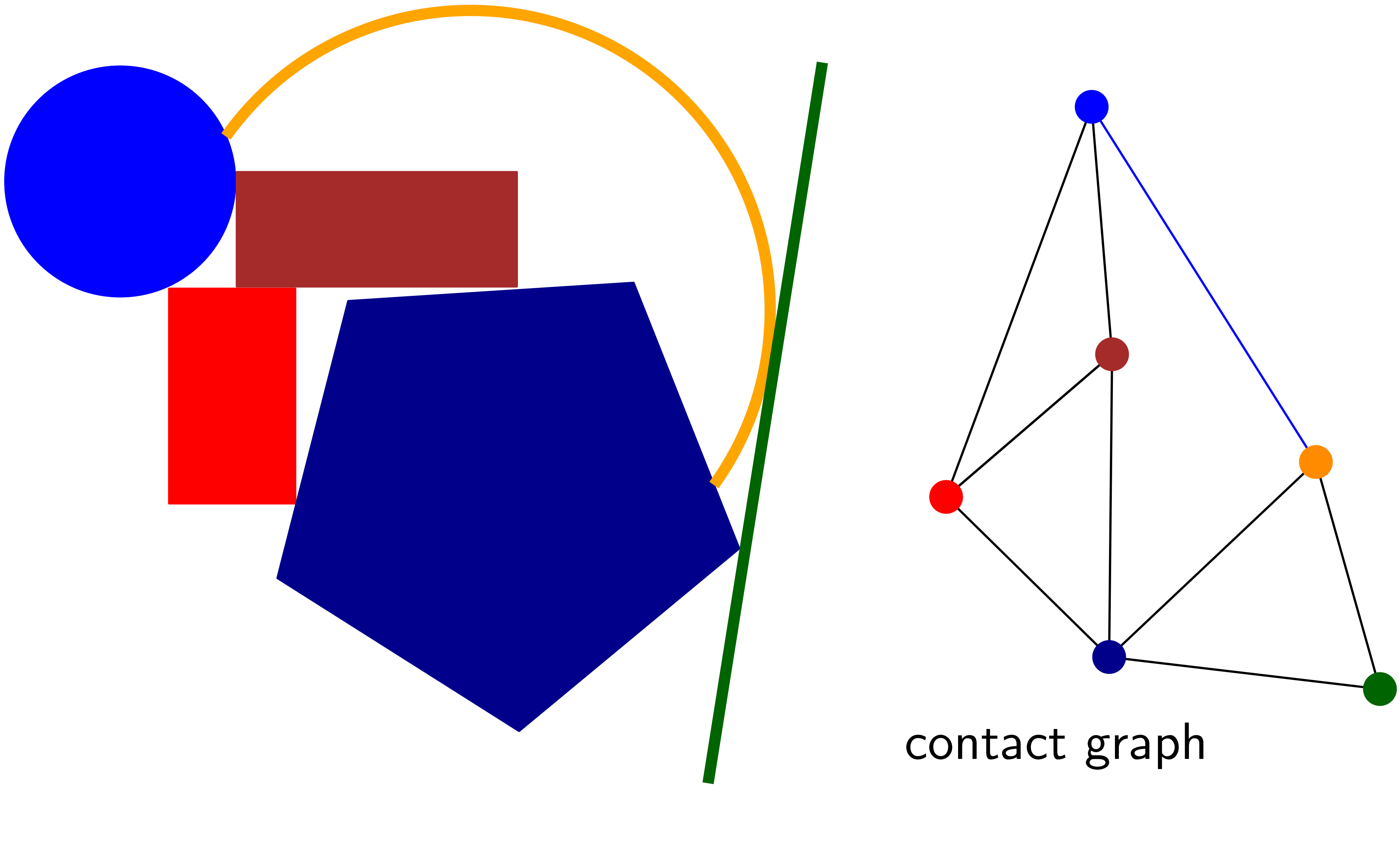

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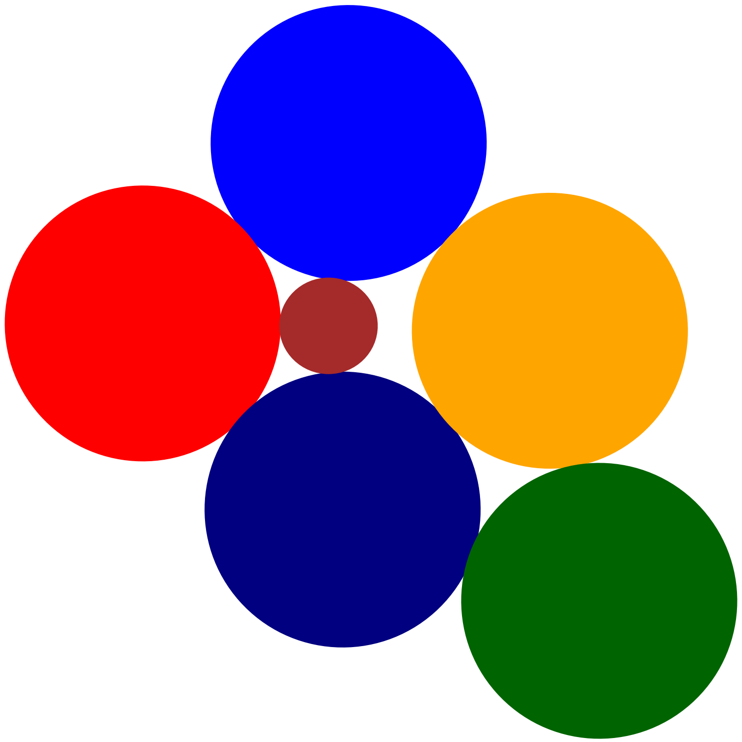
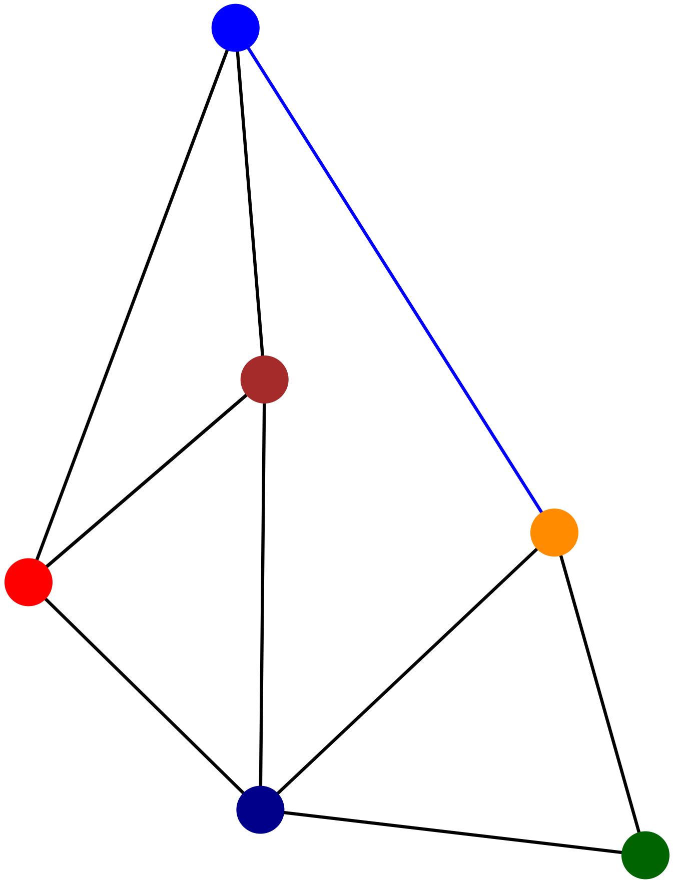


realization with disks



contact graph

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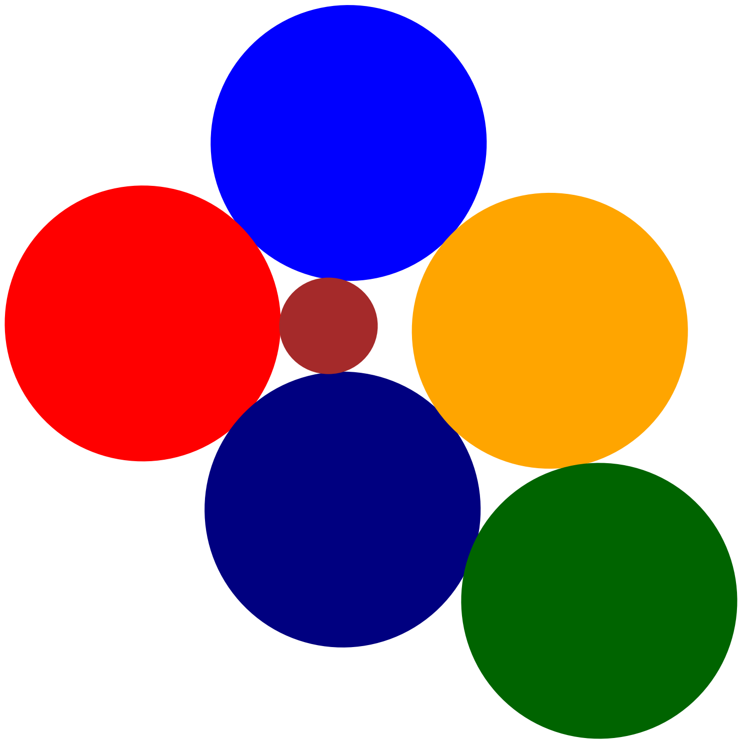

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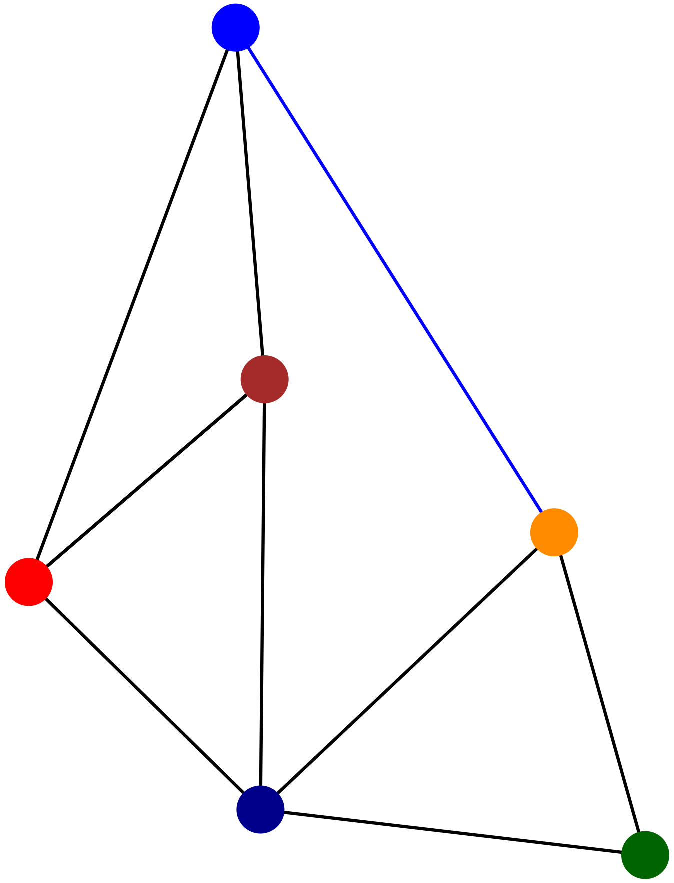
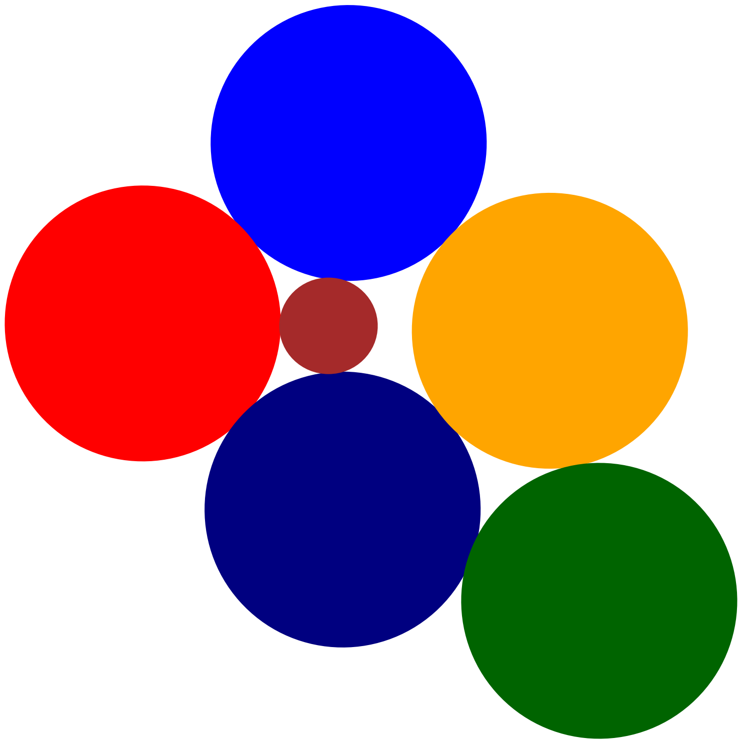


contact graph

No realization as a contact graph of **unit disks**.

Contact Graphs



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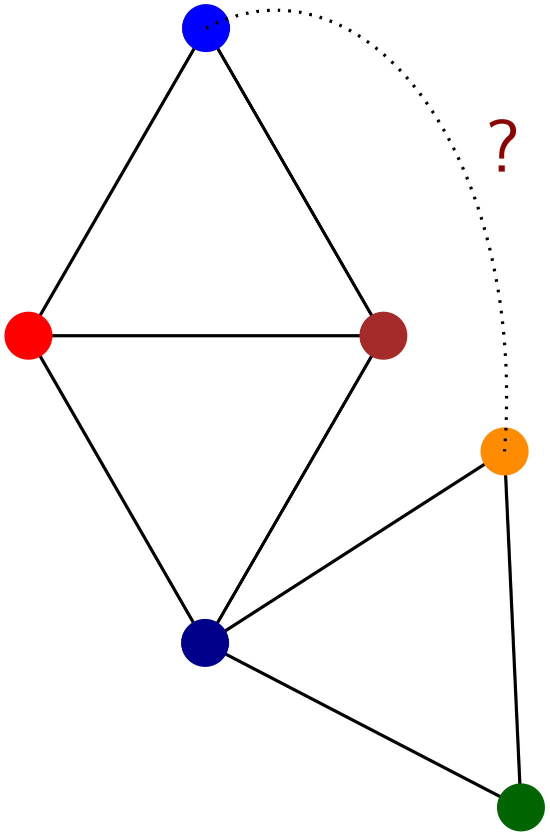
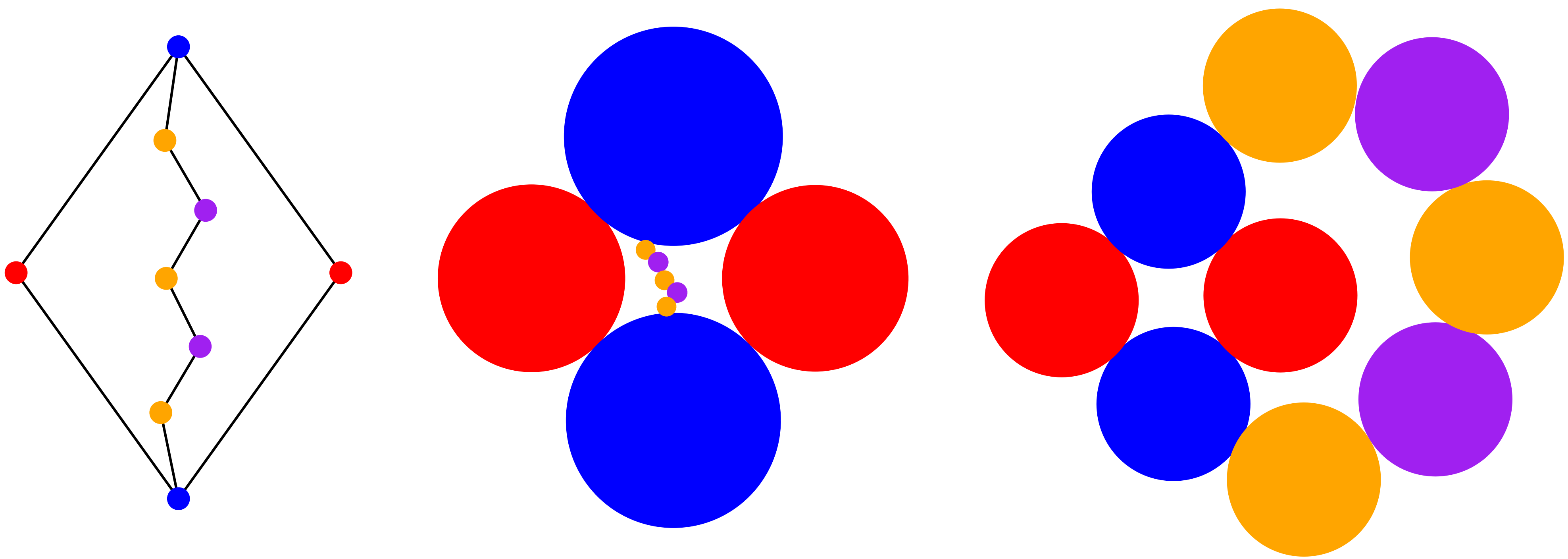


No realization as a contact graph of **unit disks**.

Recognition of Contact Graphs

Problem: Given a graph $G = (V, E)$, and
a family $\mathcal{F}(v)$ of geometric objects for every $v \in V$,
decide whether $\forall v \in V\ \exists a(v) \in \mathcal{F}(v)$:
G is the contact graph of $\{a(v) : v \in V\}$.
Two variants in $\mathbb{R}^2$:

- G is a planar graph;
- G is a plane gaph (fixed combinatorial embedding).



Contact Graphs of Disks vs Unit Disks

Koebe (1936): A graph G is a contact graph of disks iff G is planar.

Breu & Kirkpatrick (1998): NP-hard to recognize unit disk contact graphs.

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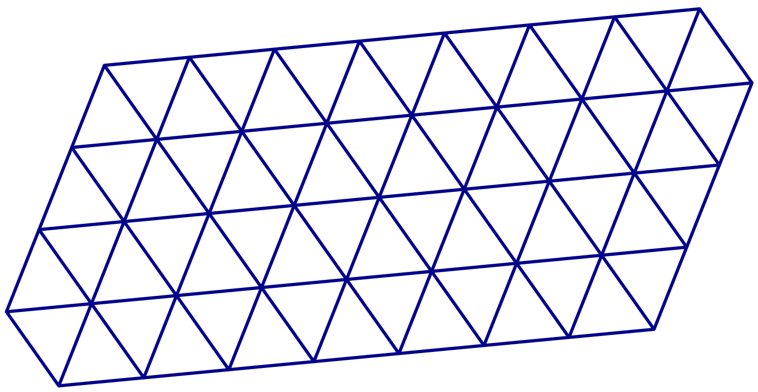

Key observation:

triangular grid has unique realization
up to rigid transformations;
can "simulate" rigid bodies.

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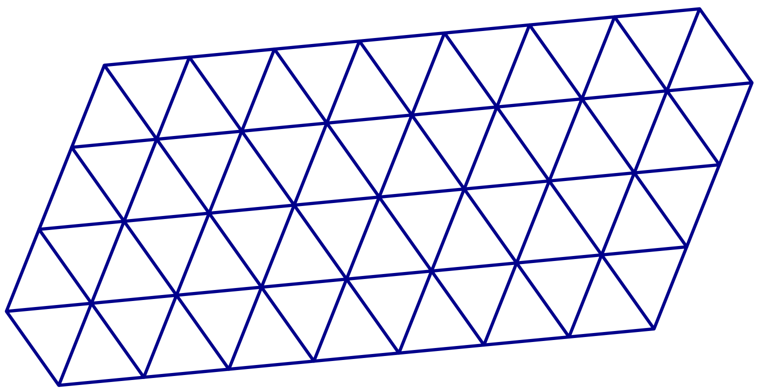
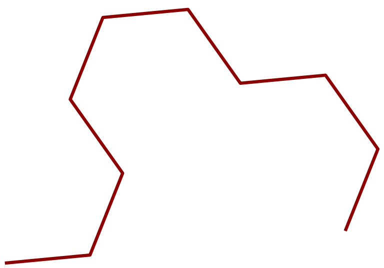




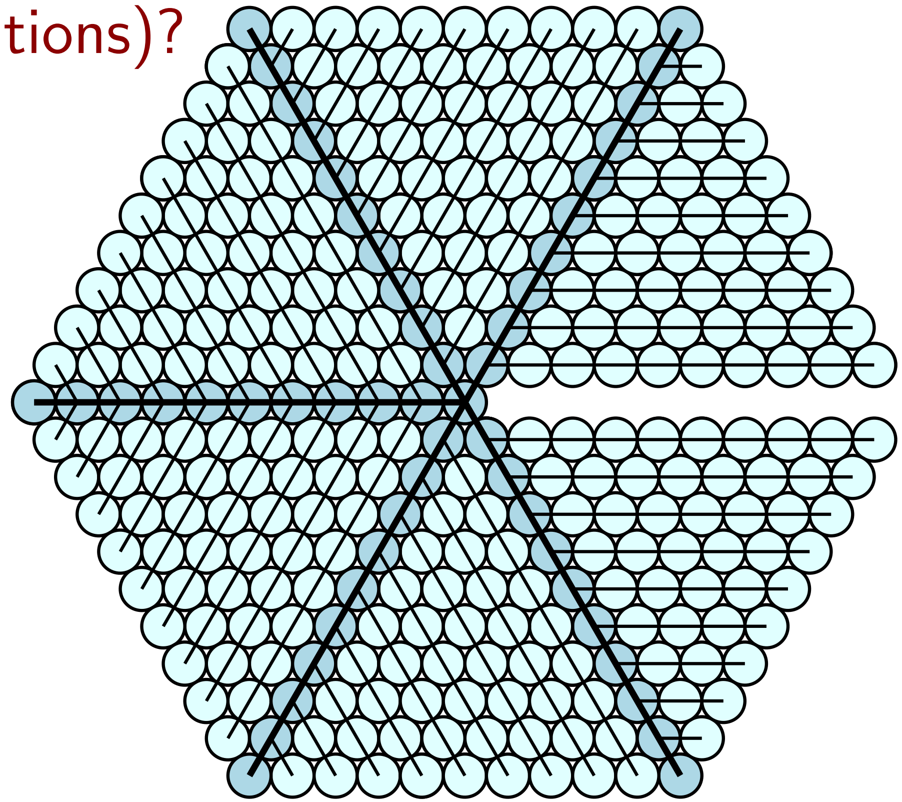
Key observation:
triangular grid has unique realization
up to rigid transformations;
can "simulate" rigid bodies.

A path is a unit disk contact graph.
The realization space is connected;
and has a high degree of freedom.

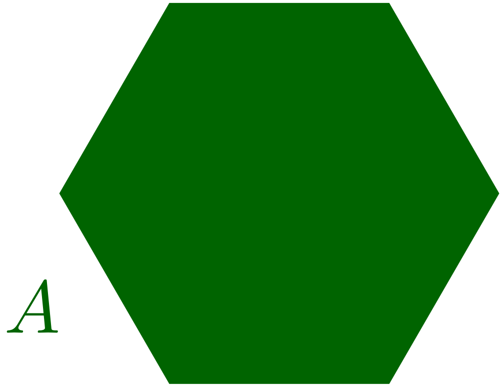
Are cycles essential for "rigidity?"
Can the realizations of a **tree** simulate a rigid body?

"Rigid" Unit Disk Contact Graphs?

Question: Is there an infinite family of **unit disk contact trees** such that in every realization, every vertex has the same position up to $O(1)$ error (modulo rigid transformations)?

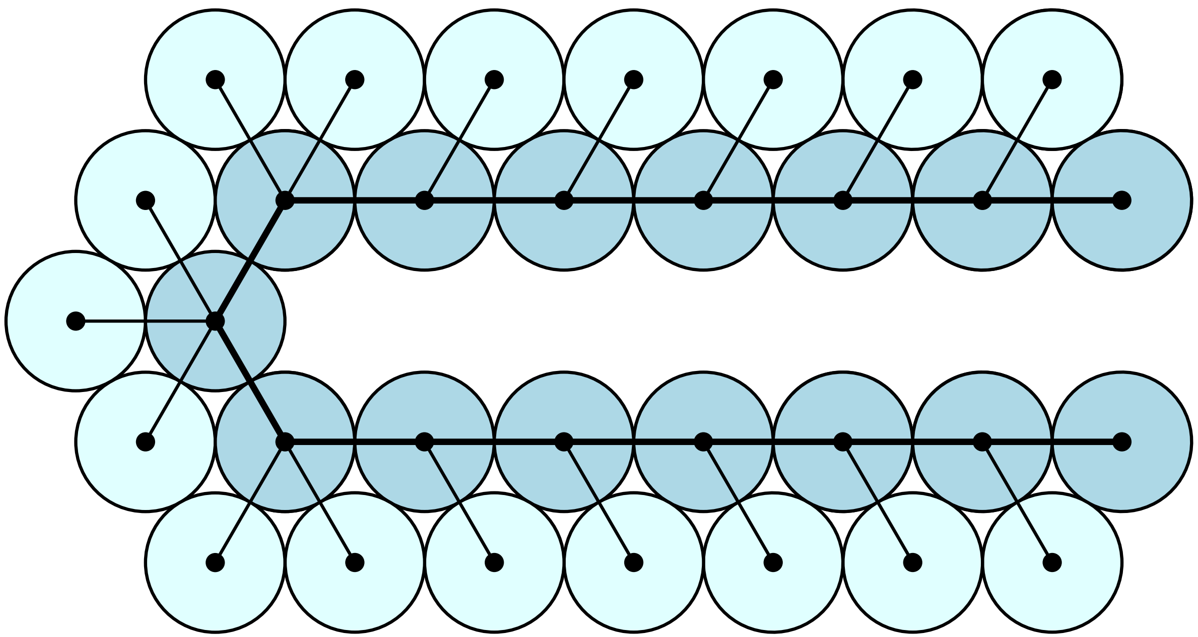


Question: Is there a rigid body $A \subset \mathbb{R}^2$ such that for every $\varepsilon > 0$ there is a **unit disk contact tree** whose realizations are within ε Hausdorff dist. from a scaled copy of A (modulo rigid transformations)?



“Rigid” Unit Disk Contact Graphs?

Bowen, Durocher, et al. (2015): The answer is positive for **plane trees** (a tree with fixed combinatroial embedding).



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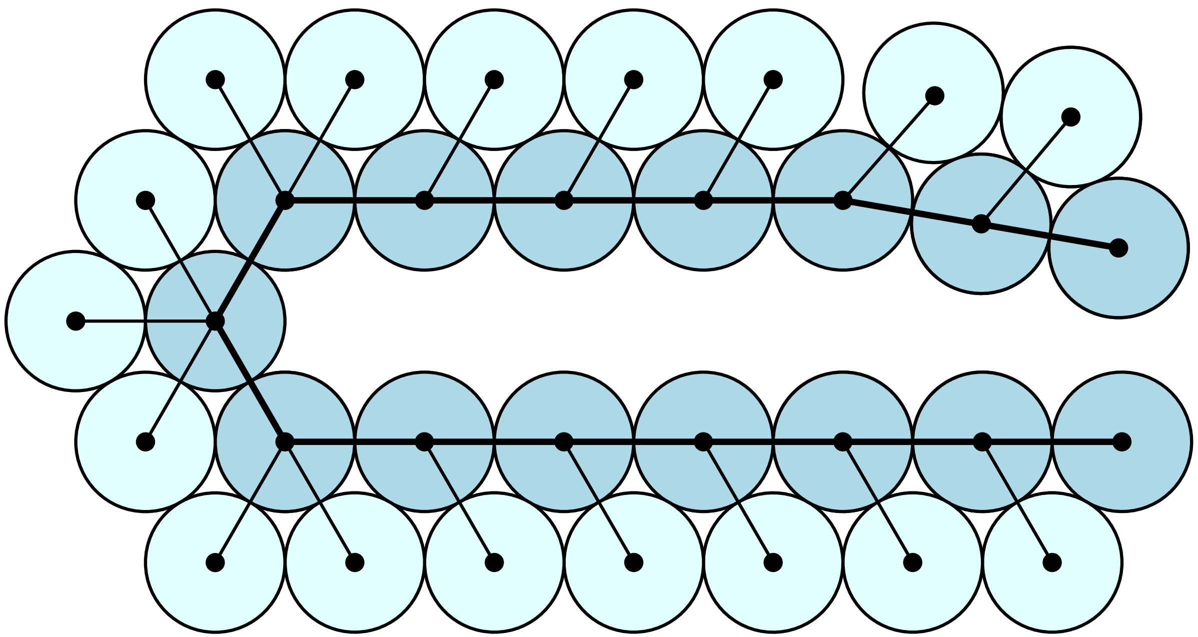

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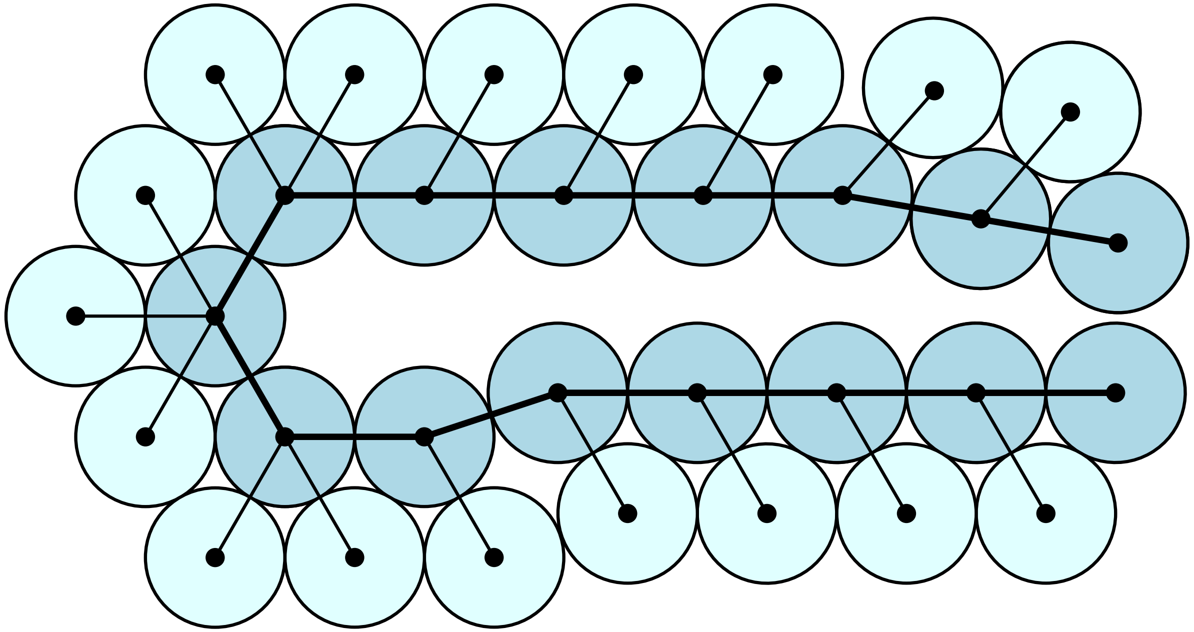

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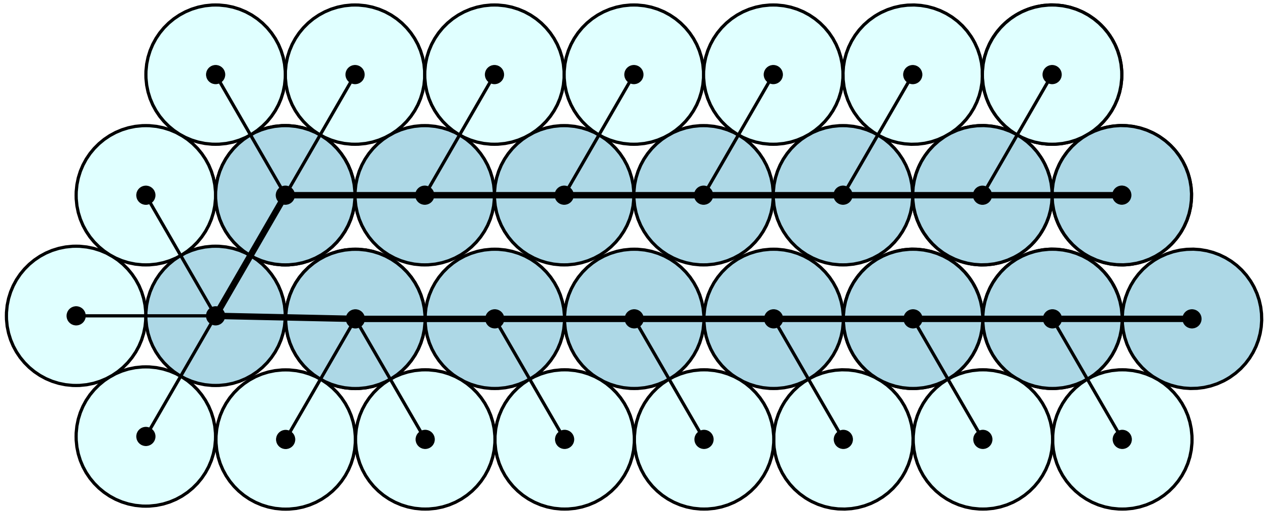

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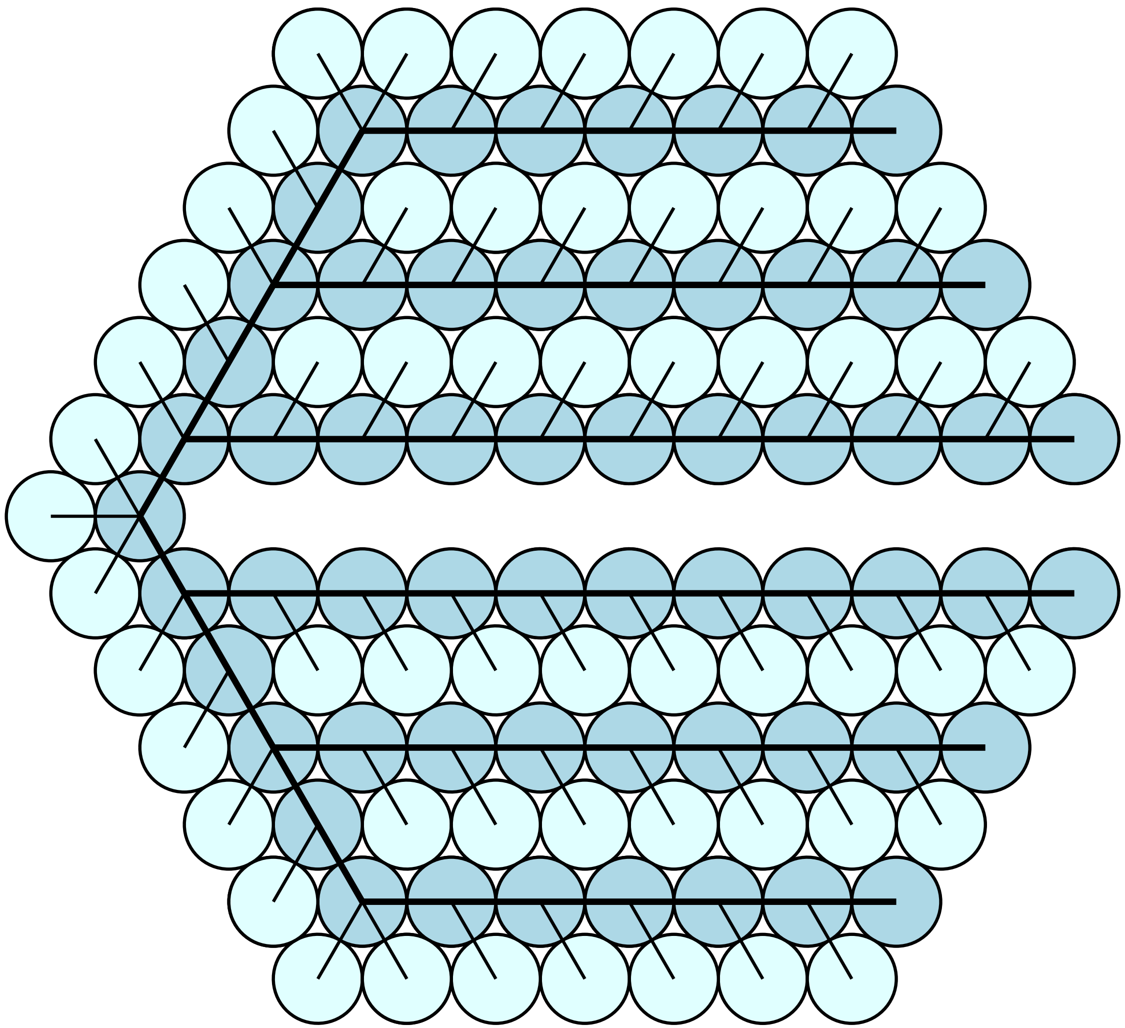

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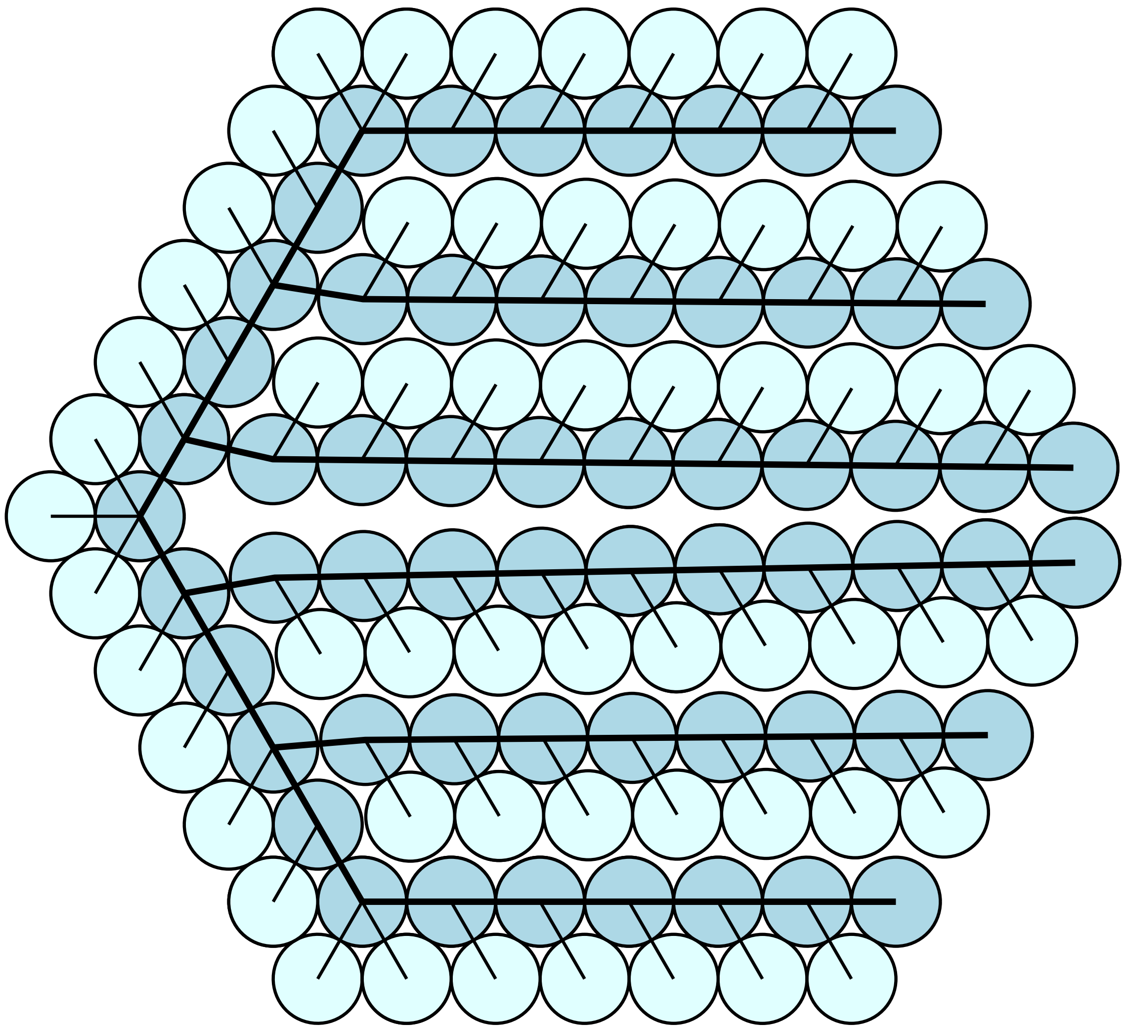

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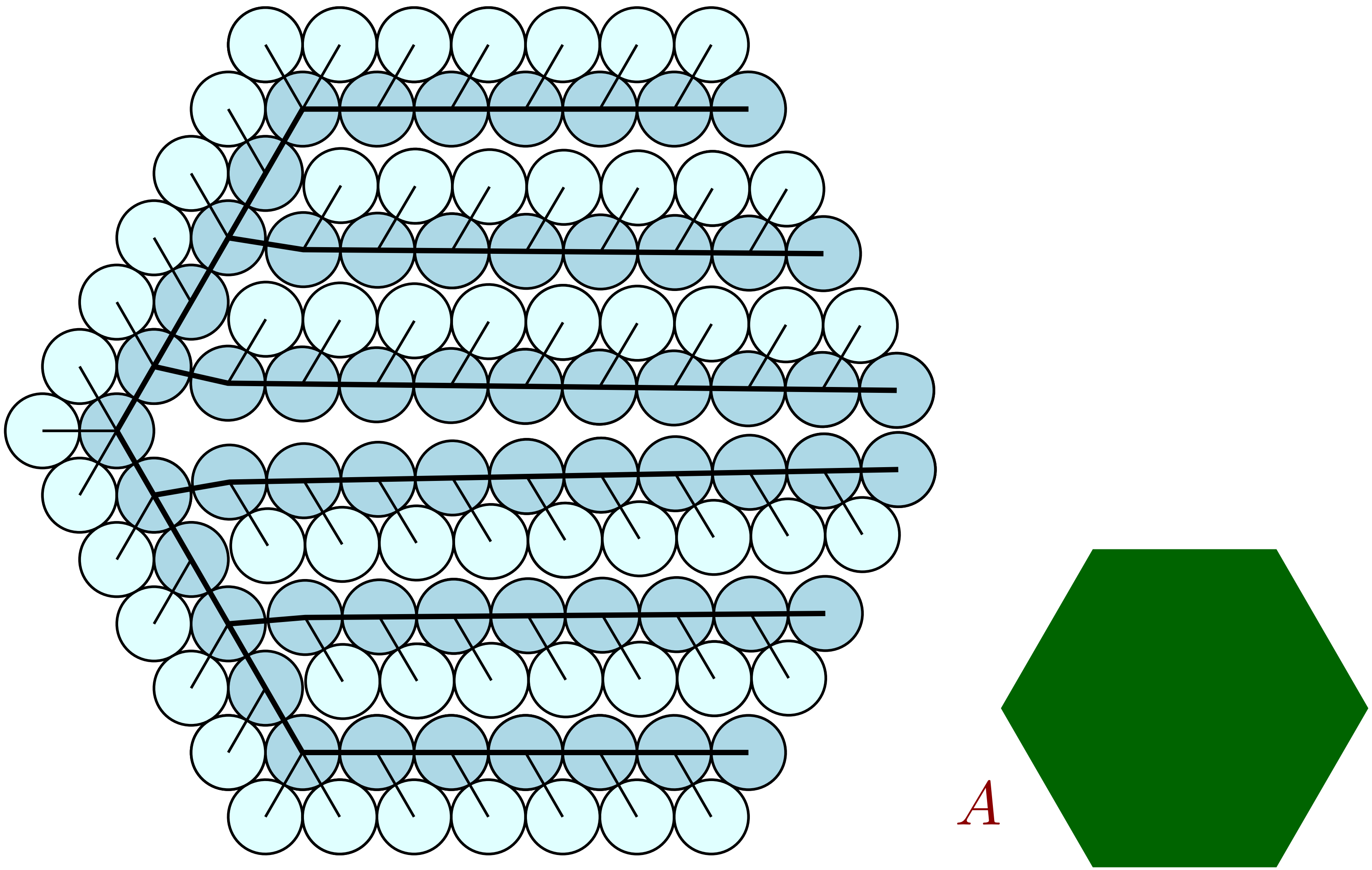
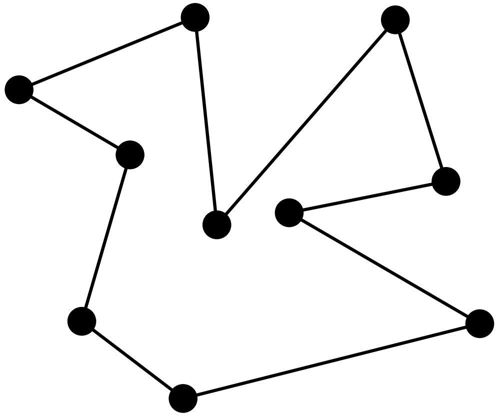
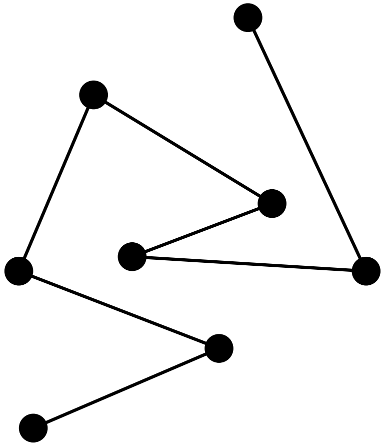


Rigidity of Linkages

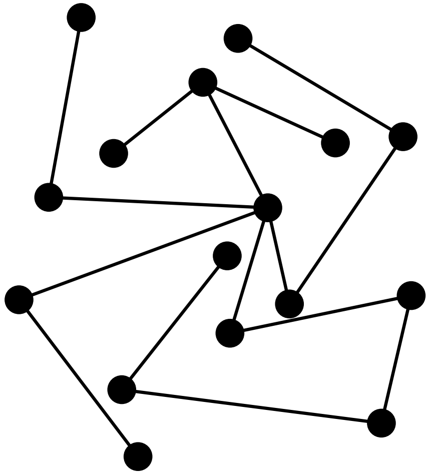
Linkage: planar straight line graph with fixed edge lengths



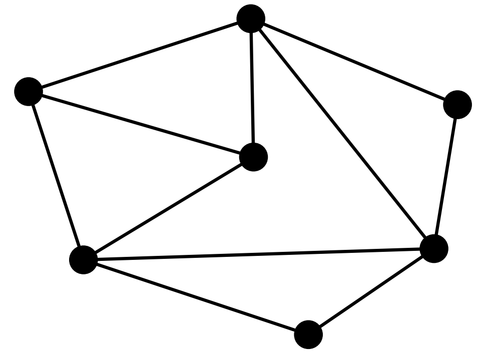
cycle



path



tree



pseudo-triangulation

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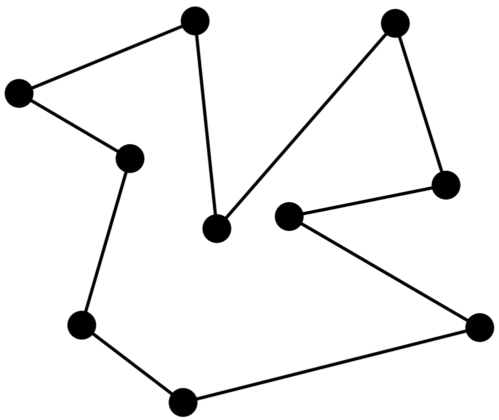

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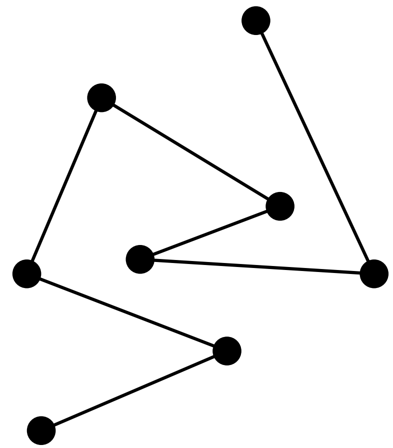
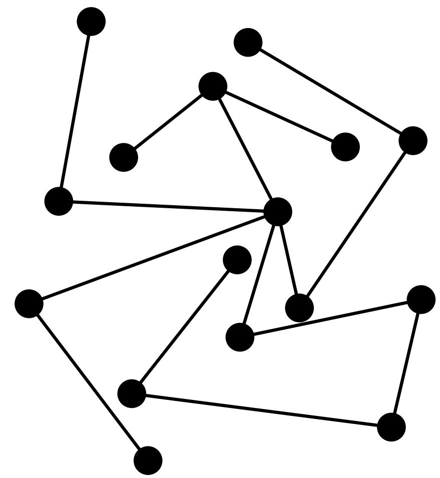


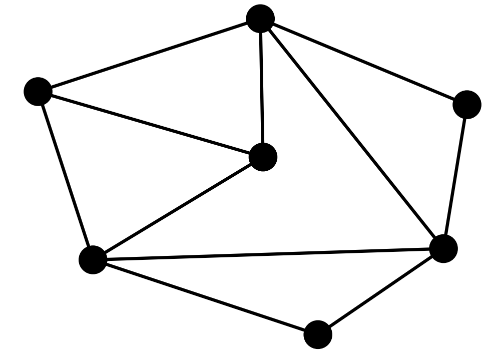
path

realization space connected
(Carpenter Rule Theroem)



tree

flat realization;
might be locked



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triangulation

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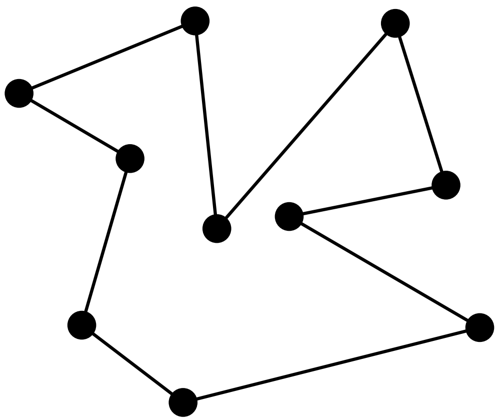

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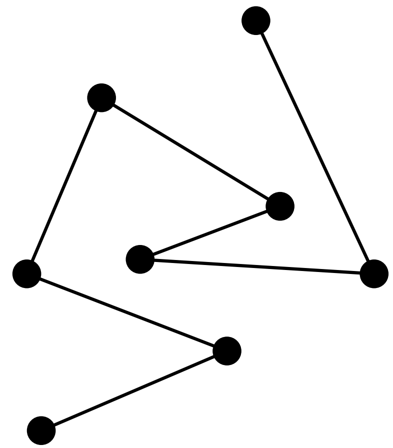

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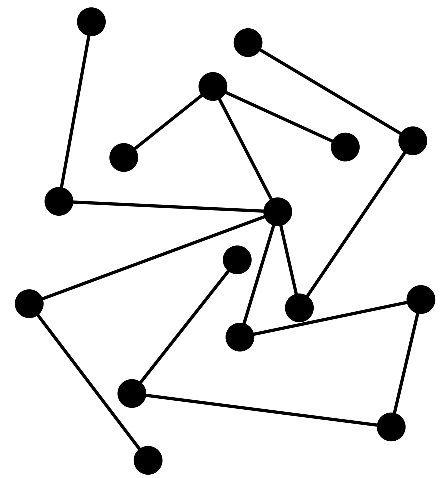

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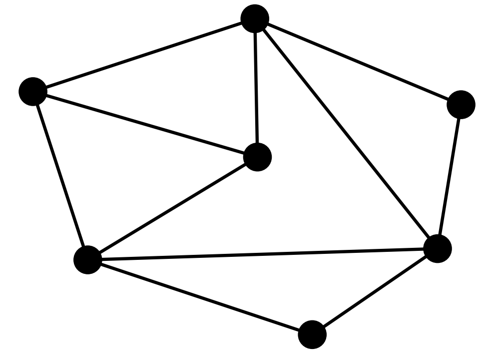

pseudo-triangulation

rigid

Laman (1970): A linkage with n joints and $2n-3$ bars in $\mathbb{R}^2$ is *generically rigid* iff every induced subgraph on k joints has at most $2k-3$ bars.

Rigidity of Linkages

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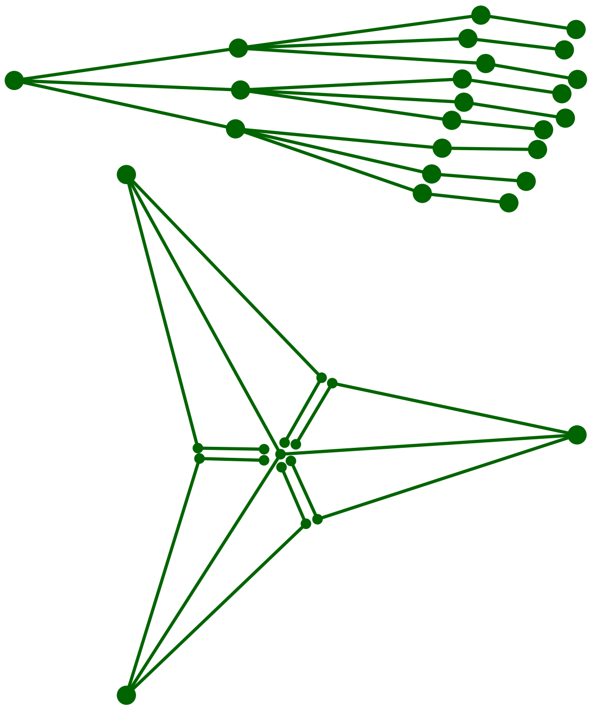
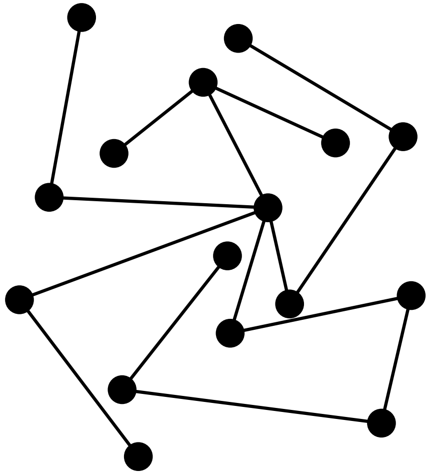


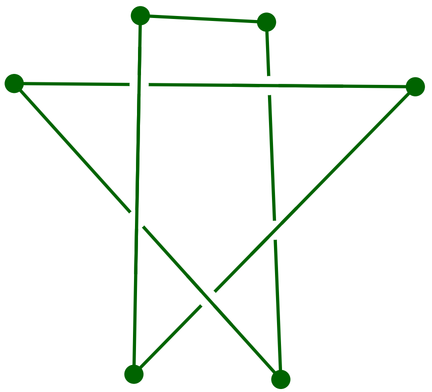
[Connelly, Demaine, & Rote, 2002]



tree

flat realization; might be locked

A path can also lock in $\mathbb{R}^3$.



[Toussaint, 1999]

Question: Does an equilateral **path** linkage have a connected configuration space in $\mathbb{R}^3$?

Question: Does a **path** have a connected realization space as a contact graph of unit disks in $\mathbb{R}^3$?

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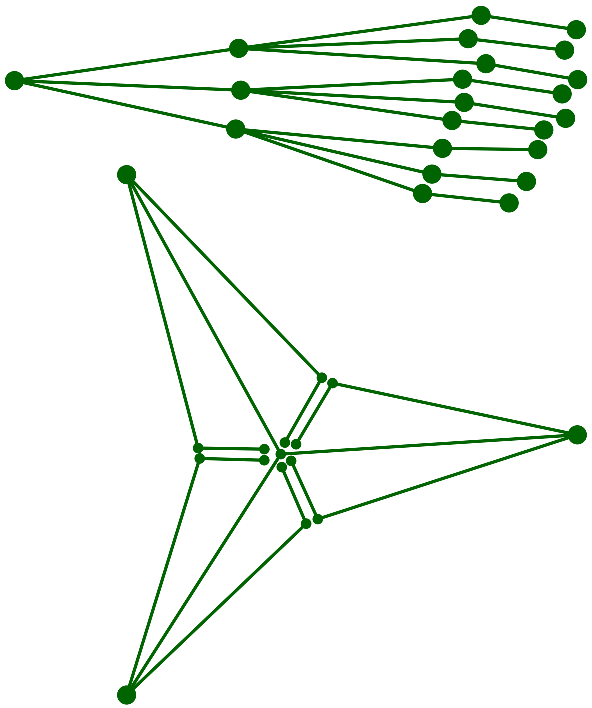

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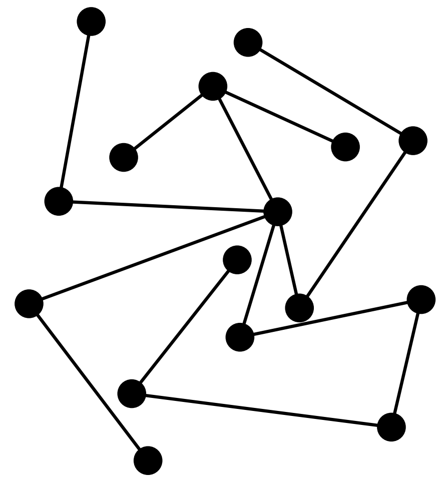

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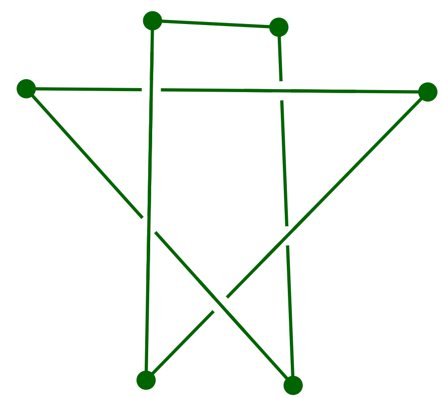

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Question: Does an equilateral **path** linkage have a connected configuration space in $\mathbb{R}^3$?

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Body-Hinge Frameworks

Linkages = Realizations of contact graphs of bars, where the point of contact is given.

Consider contact graphs on other geometric objects
with given contact points.

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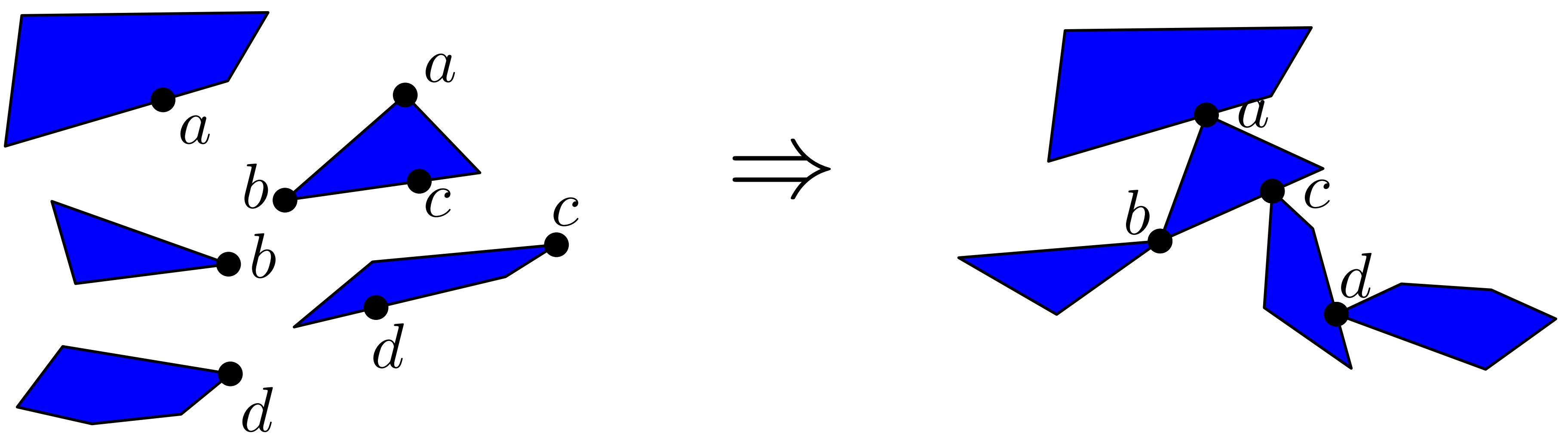

Polygonal Linkage Realizability: Given a set of polygons, a contact graph, and contact points, decide realizability.
Variants: (1) translation and rotation;
(2) translation, rotation, and reflection.

Body-Hinge Frameworks

Bowen et al. (2015):

- For **trees**, both variants are strongly NP-hard (reductions from Planar3SAT and NAE3SAT).
- For **paths**, both variants are weaky NP-hard (reductions from Partition).

Body-Hinge Frameworks

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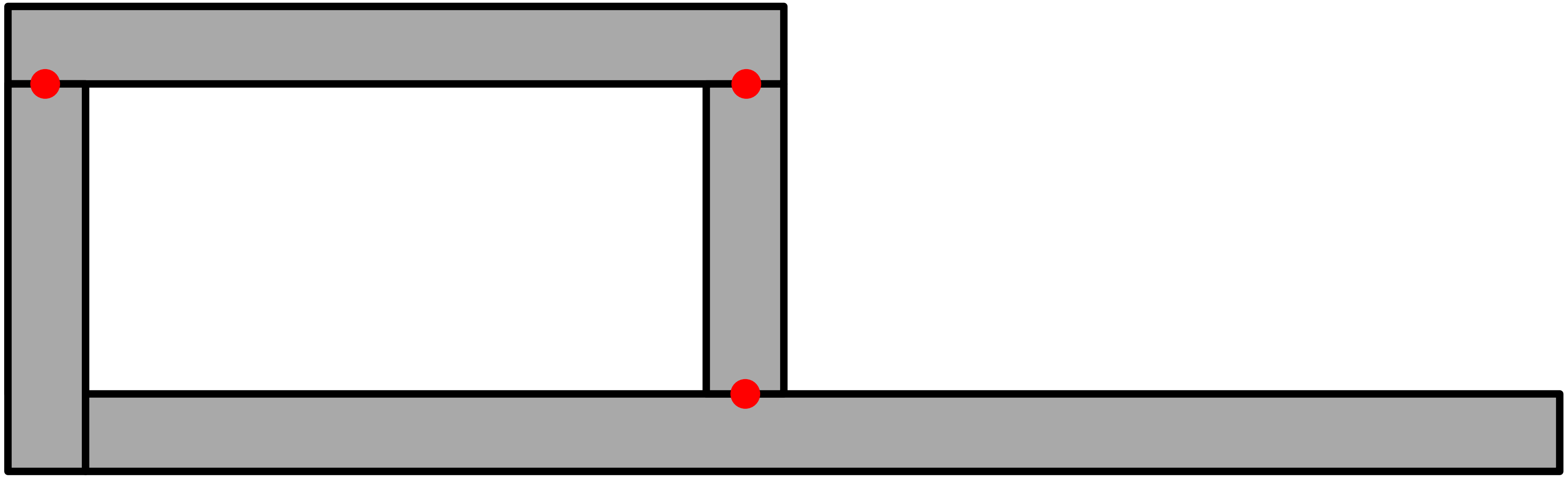

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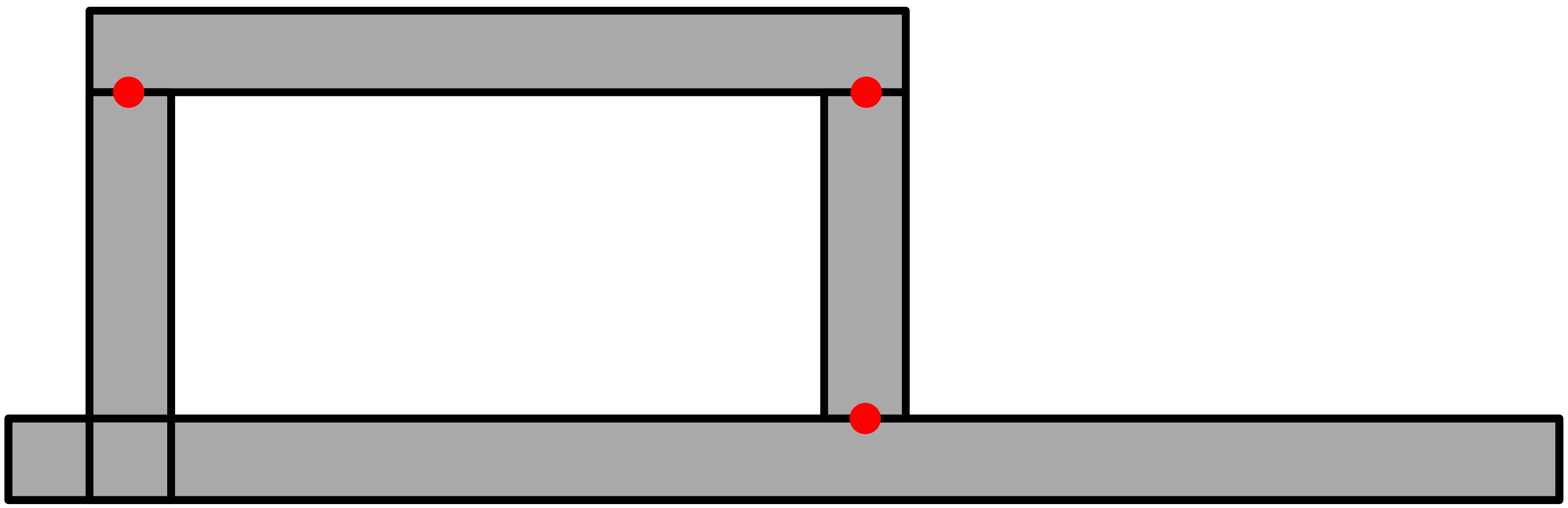

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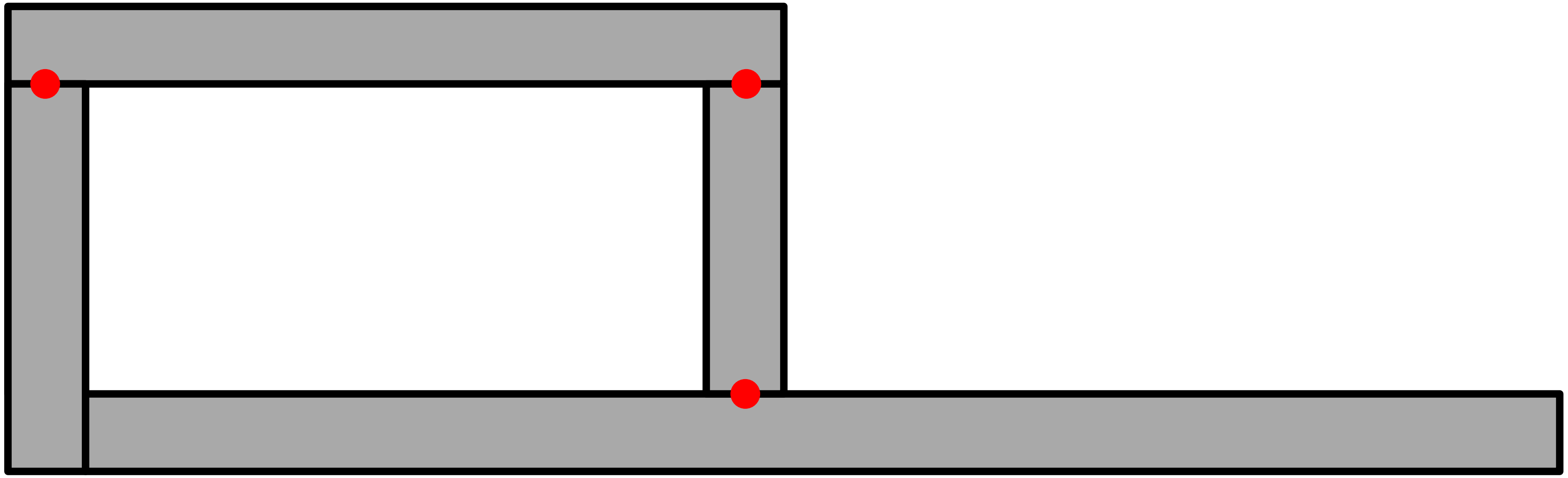

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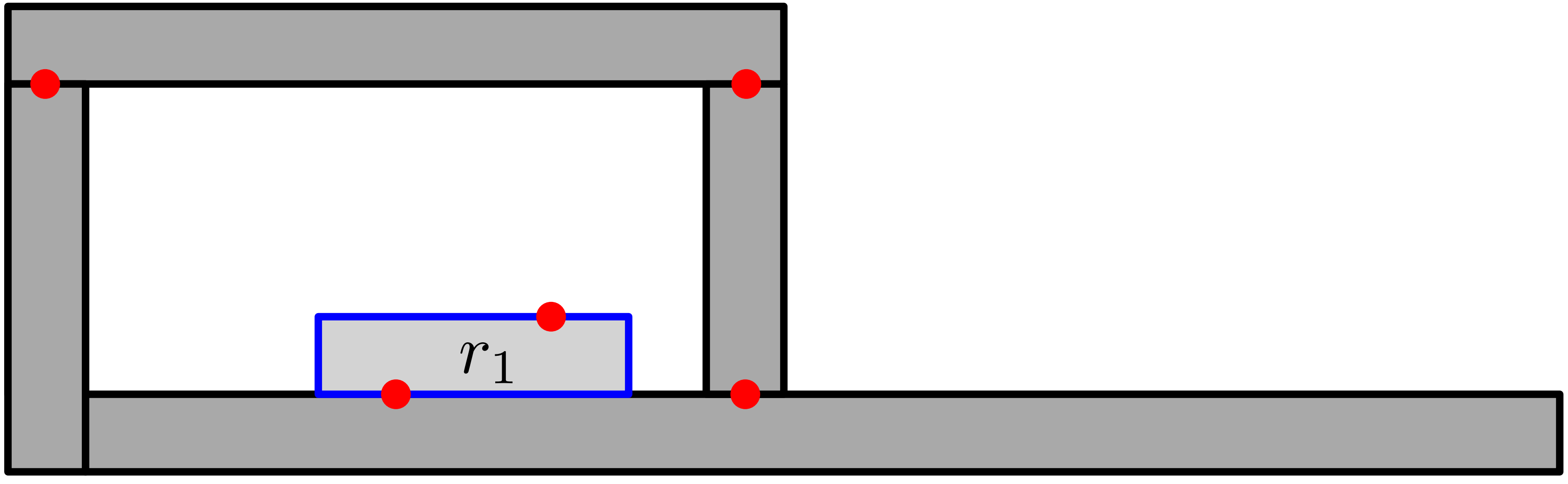

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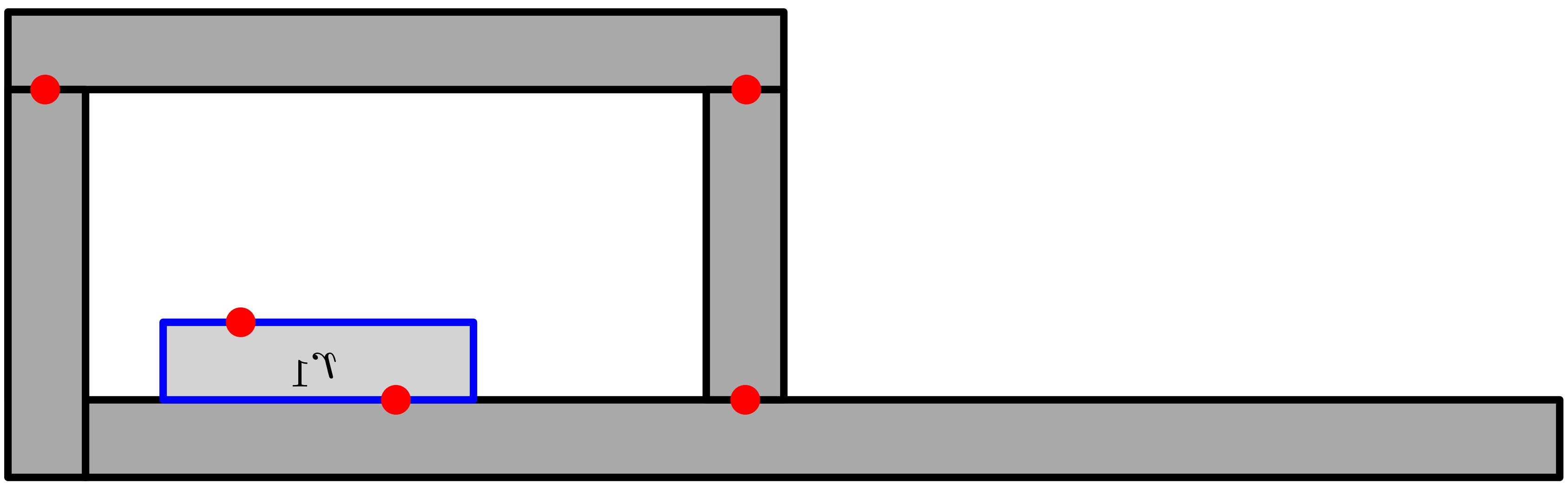

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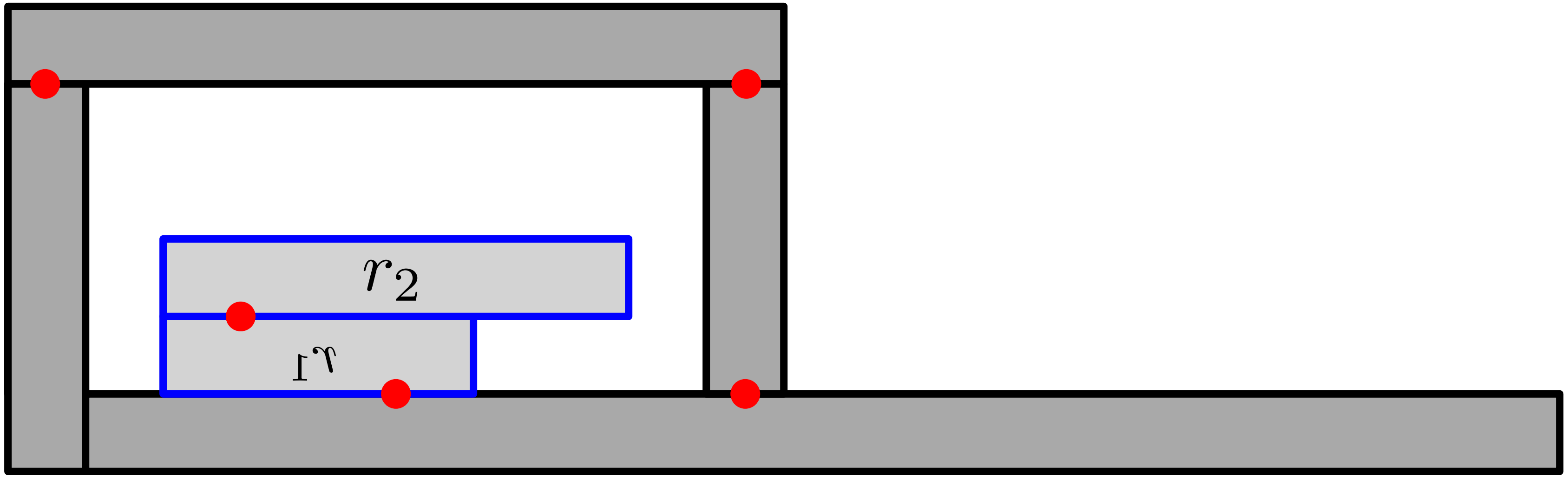

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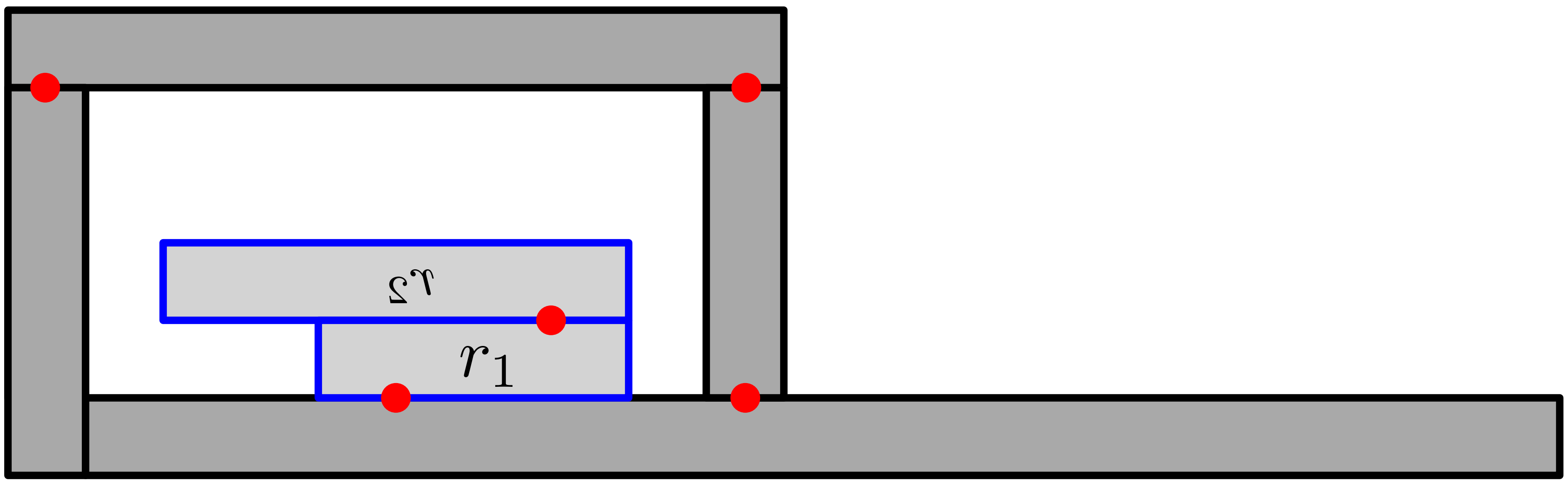

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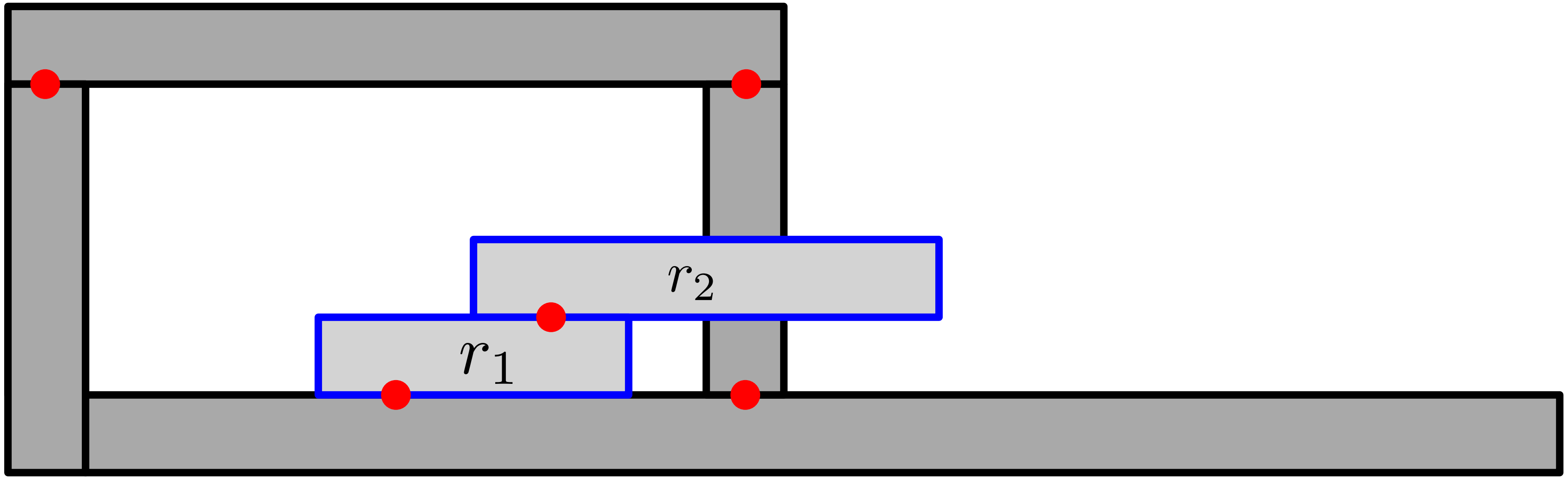

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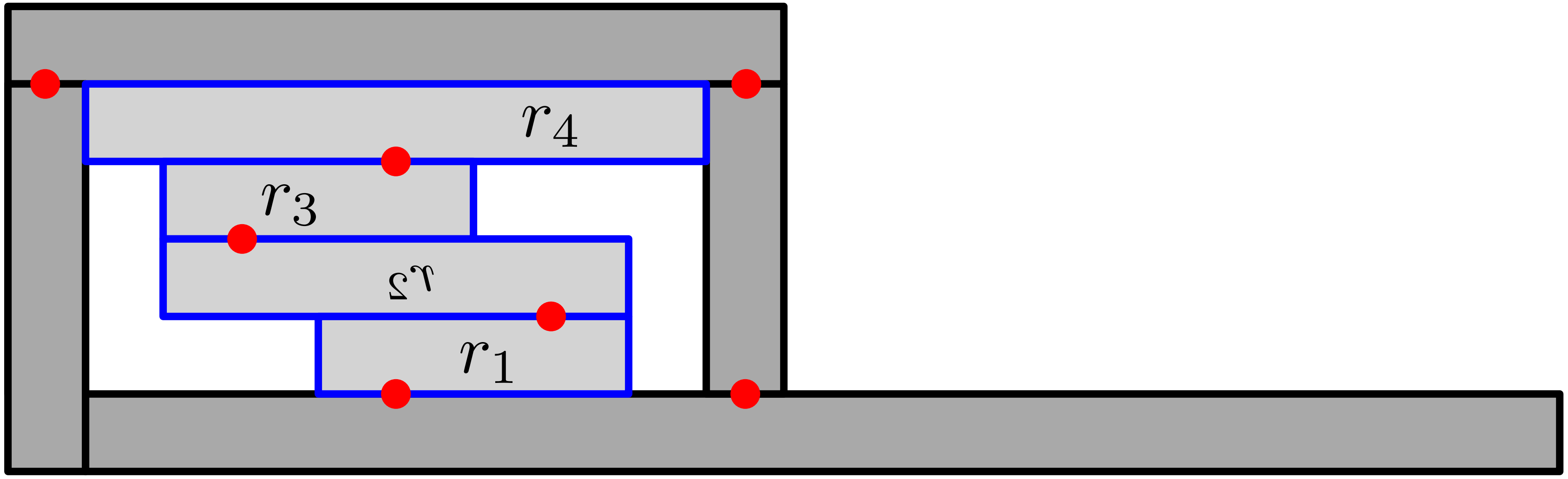

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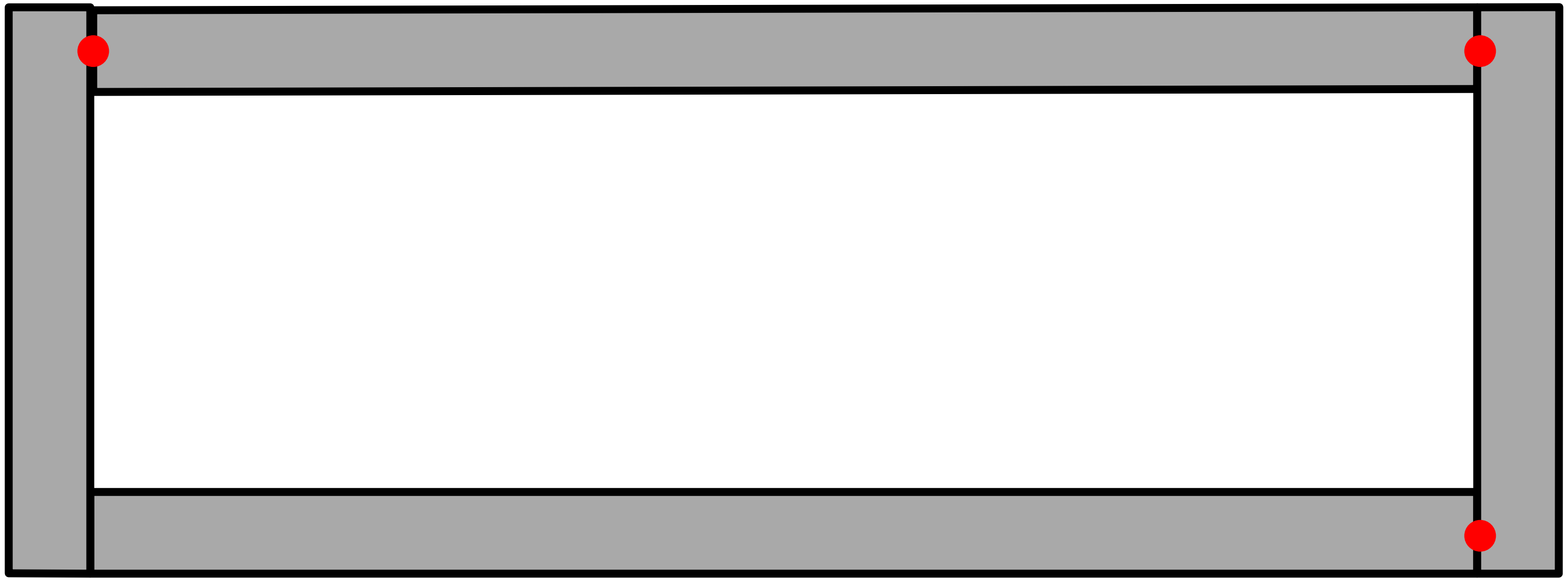

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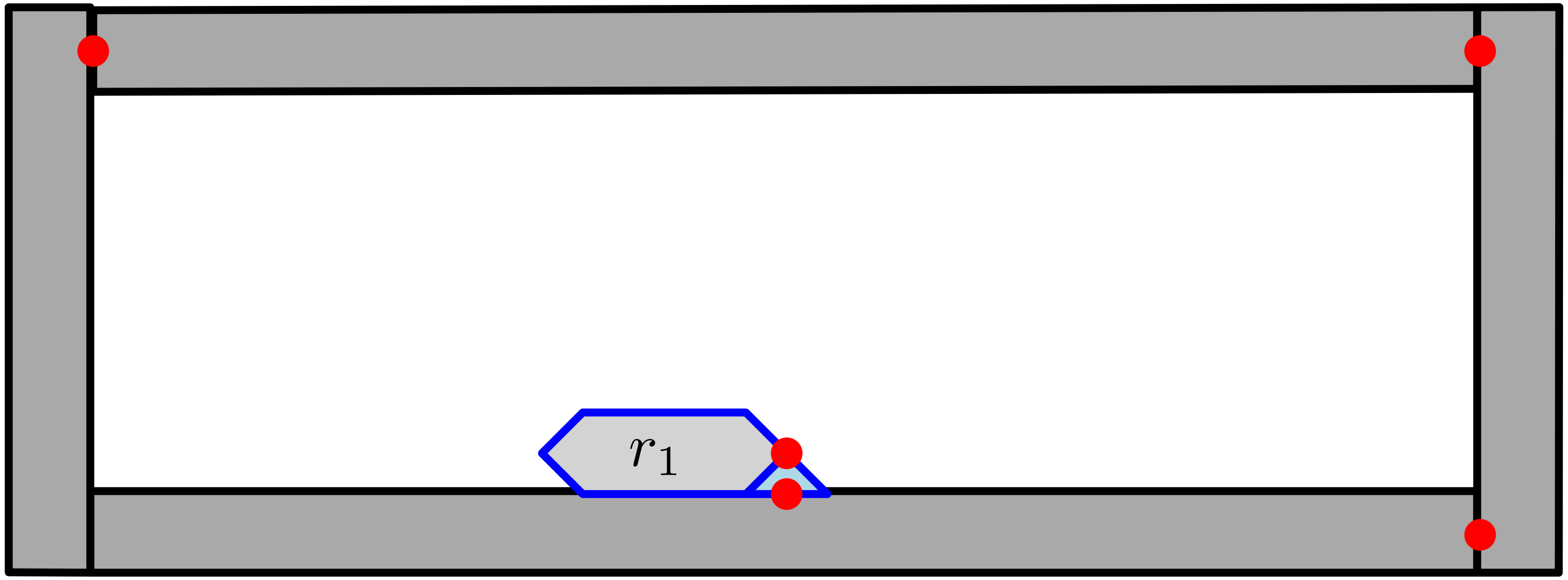

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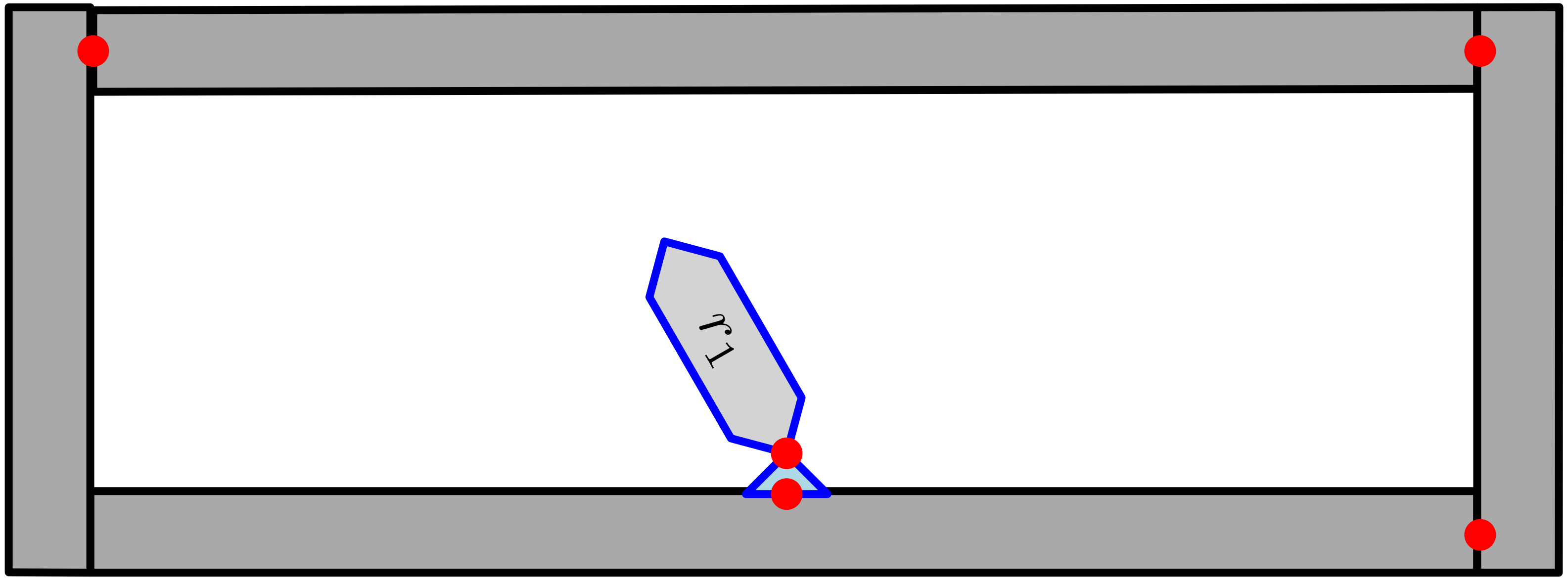

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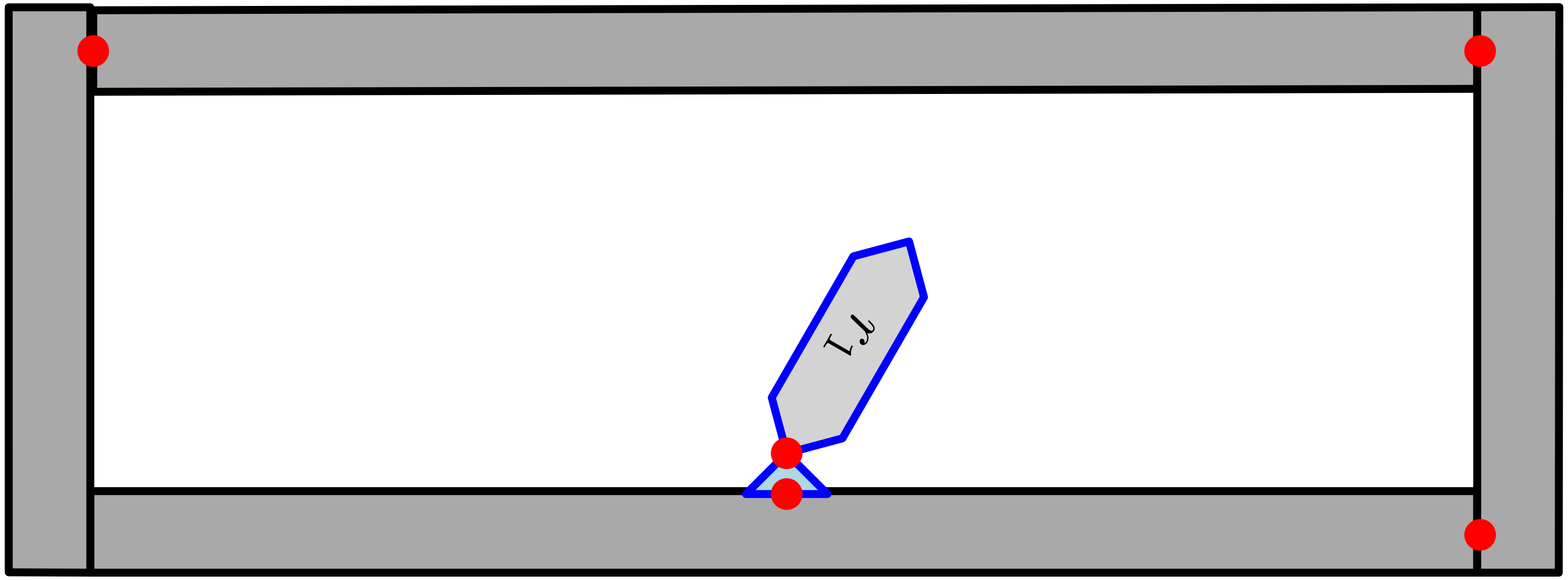

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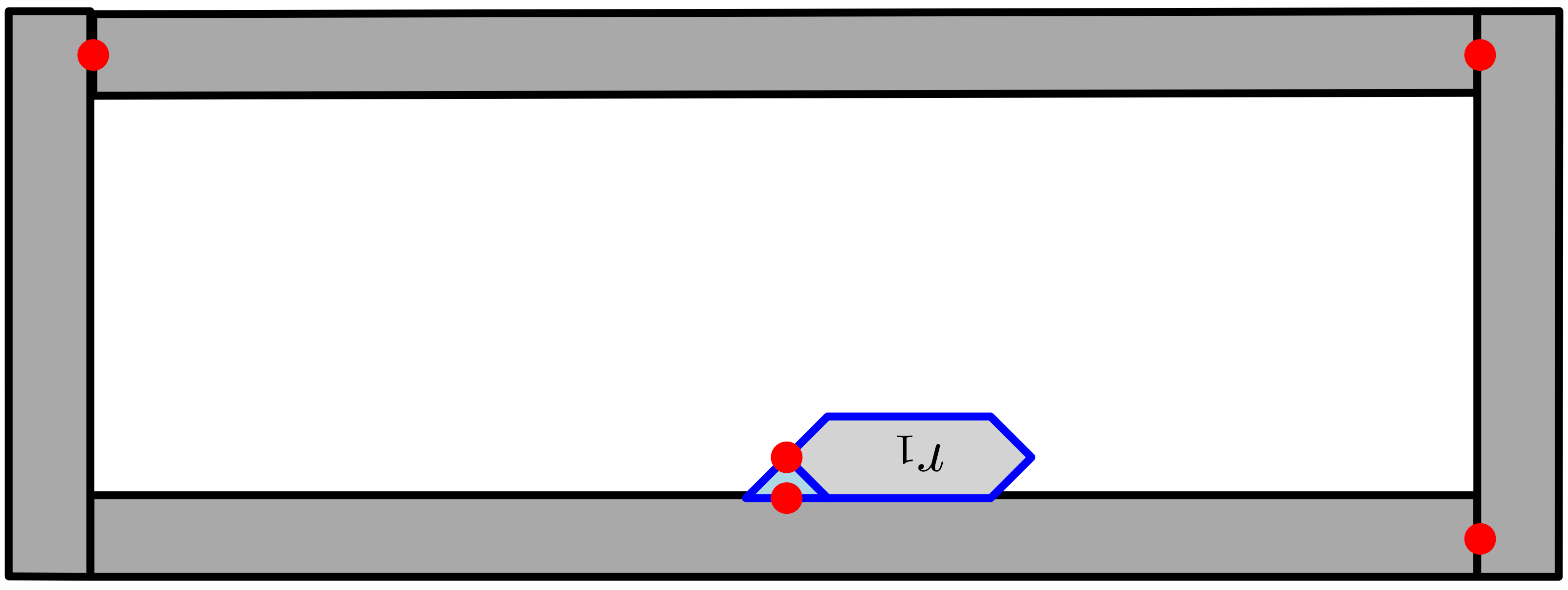

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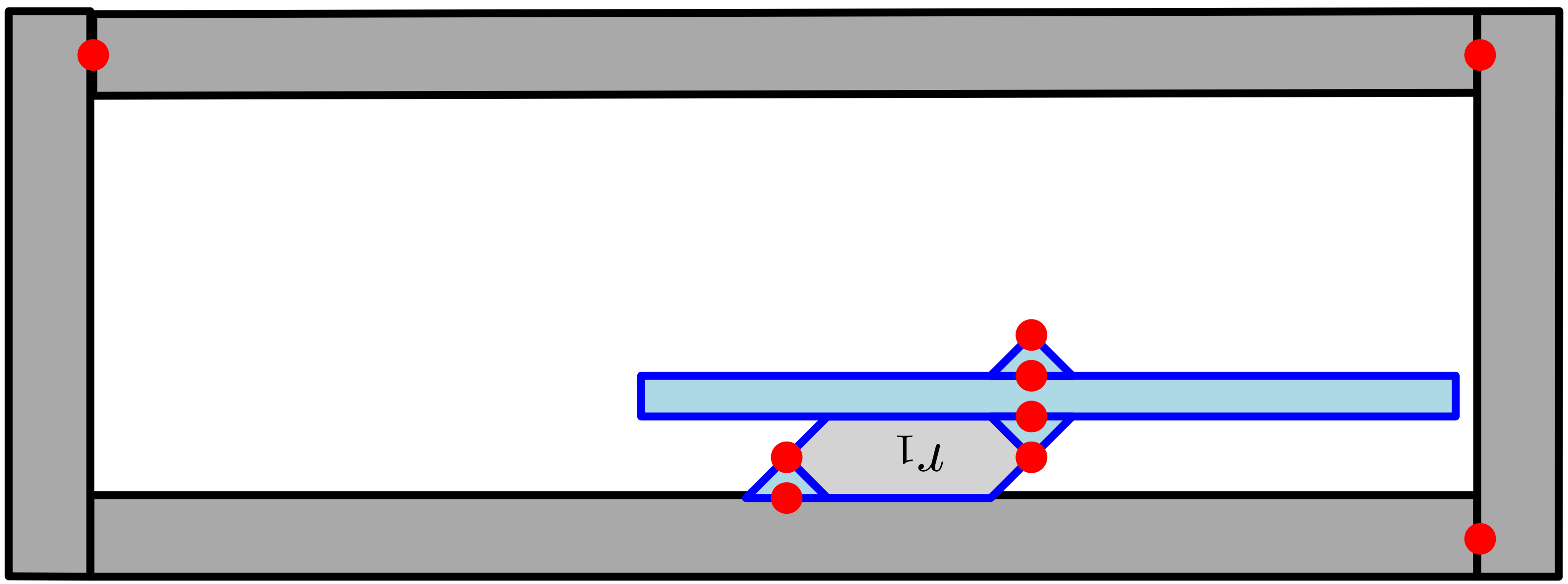

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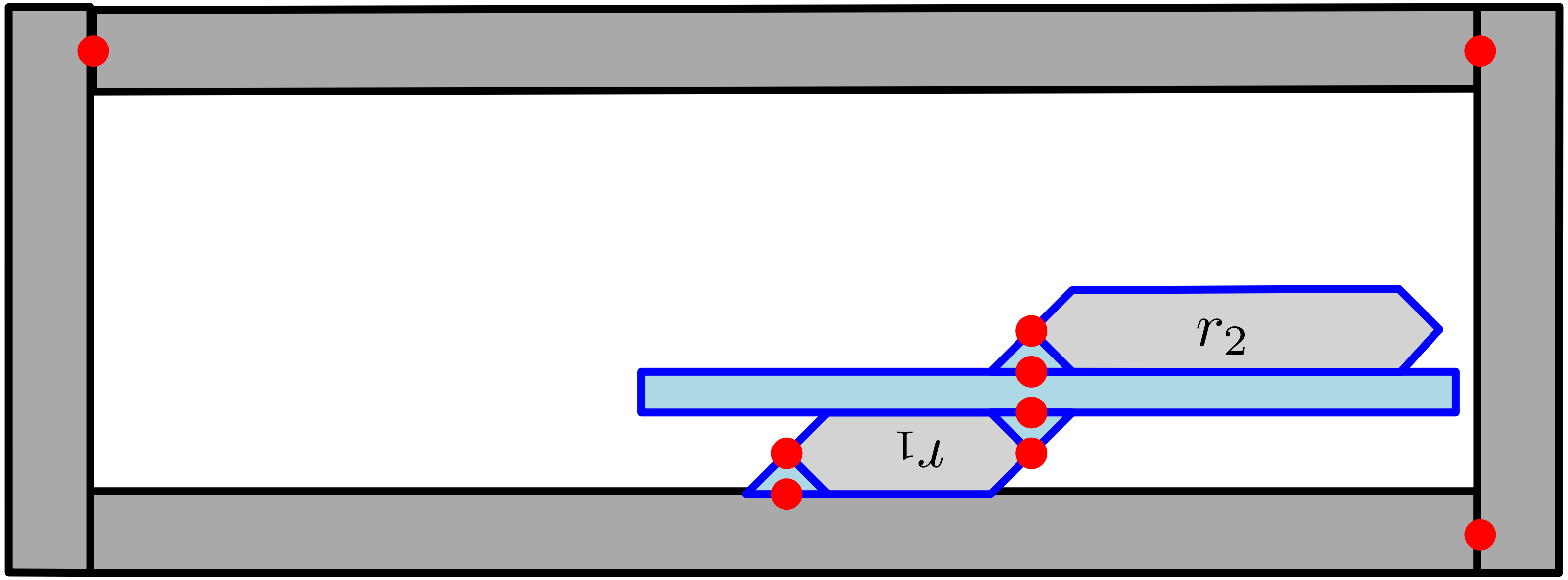

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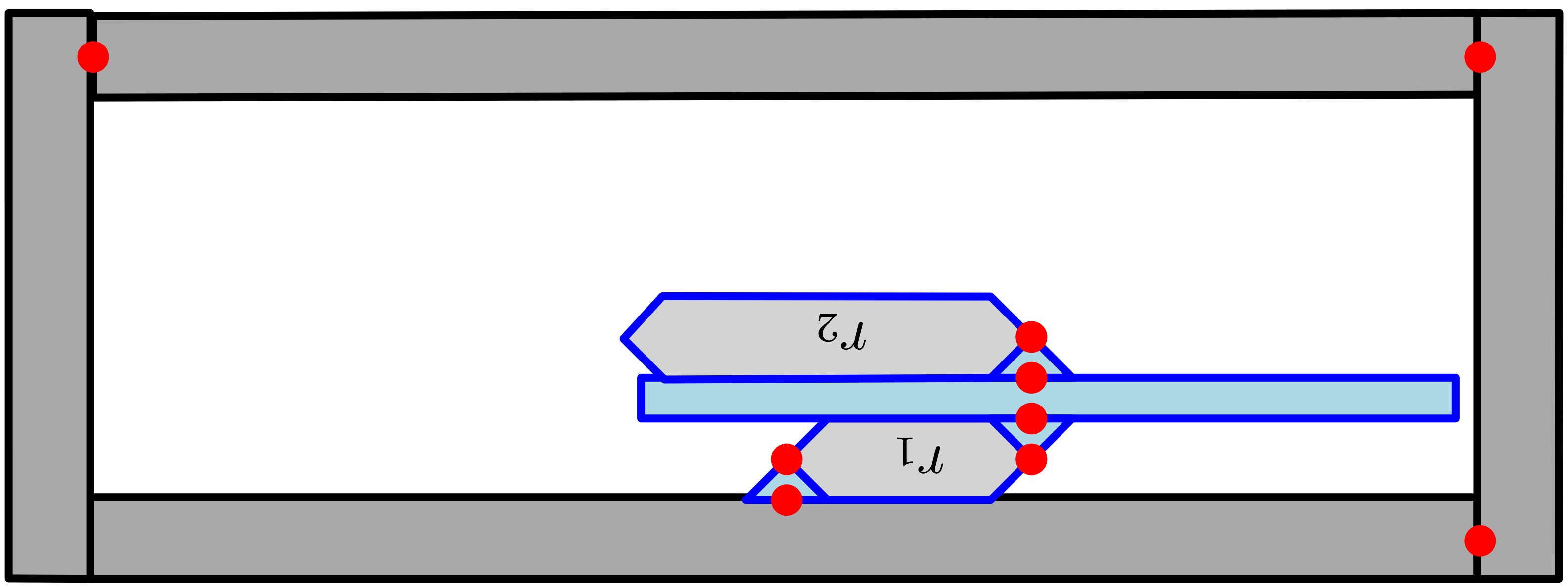

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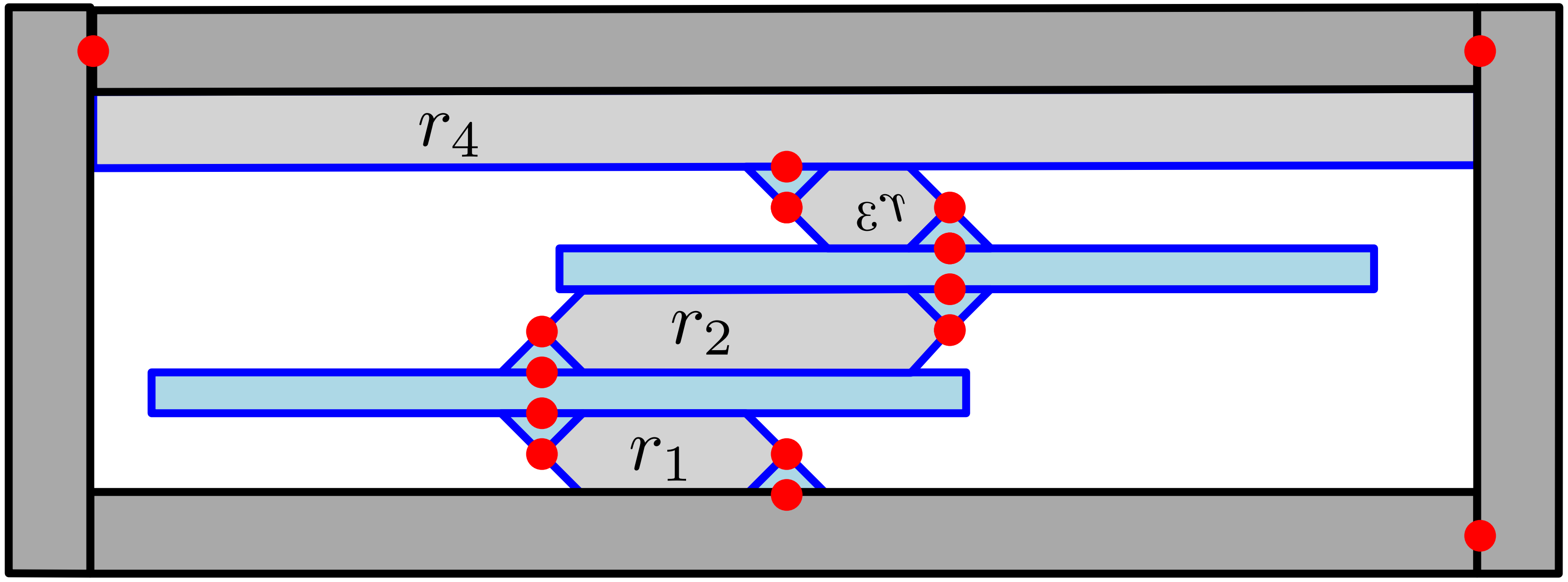
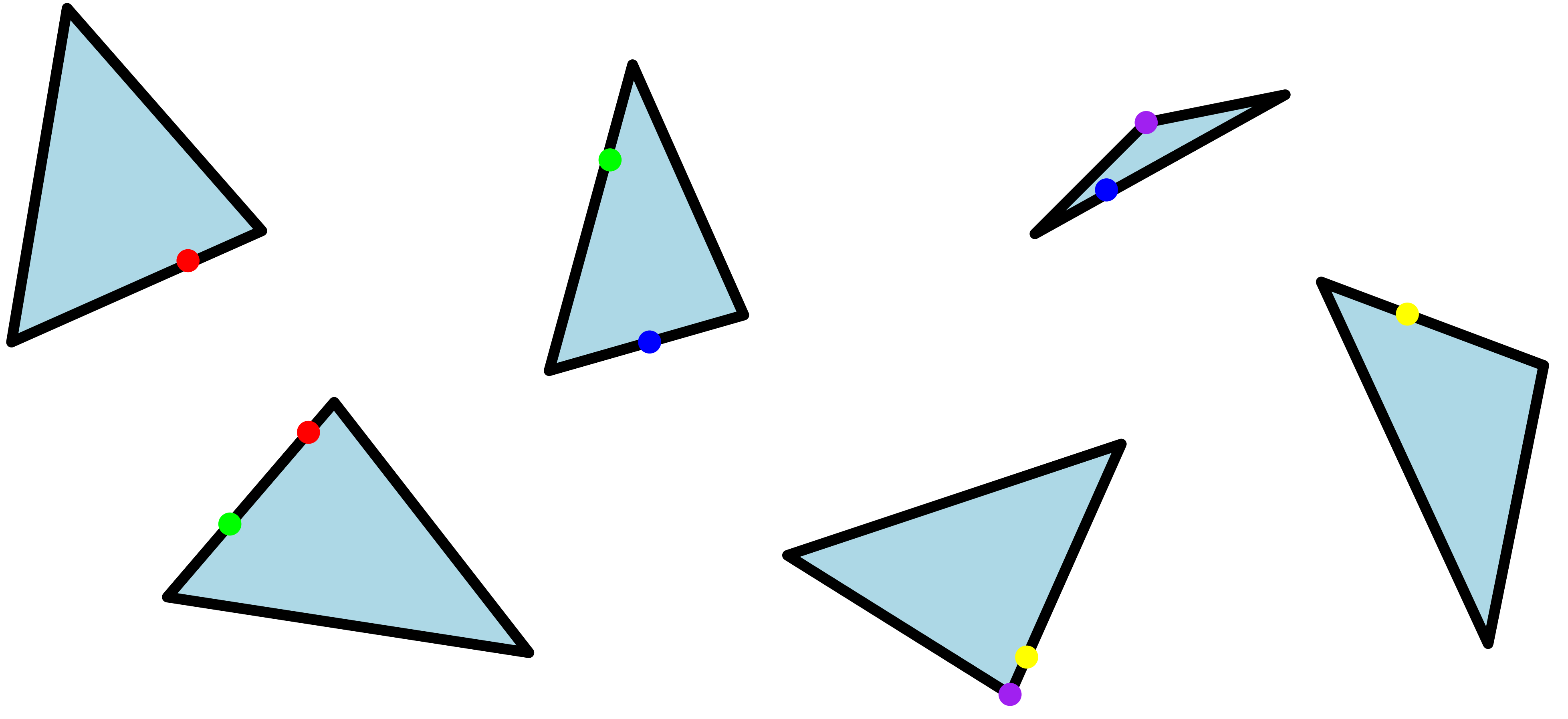


Body-Hinge Frameworks

For **paths**, both variants are weaky NP-hard even if all polygons are triangles.



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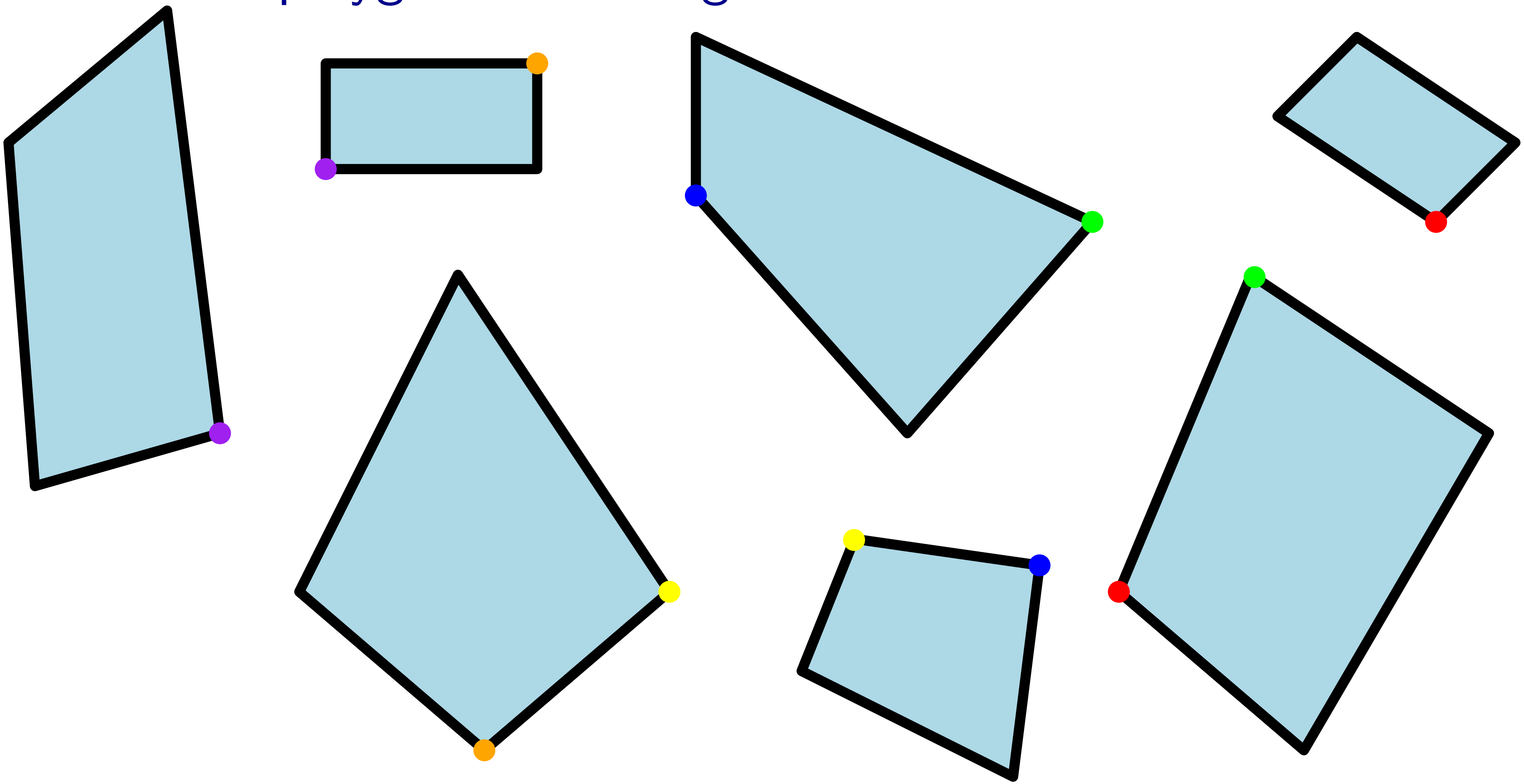

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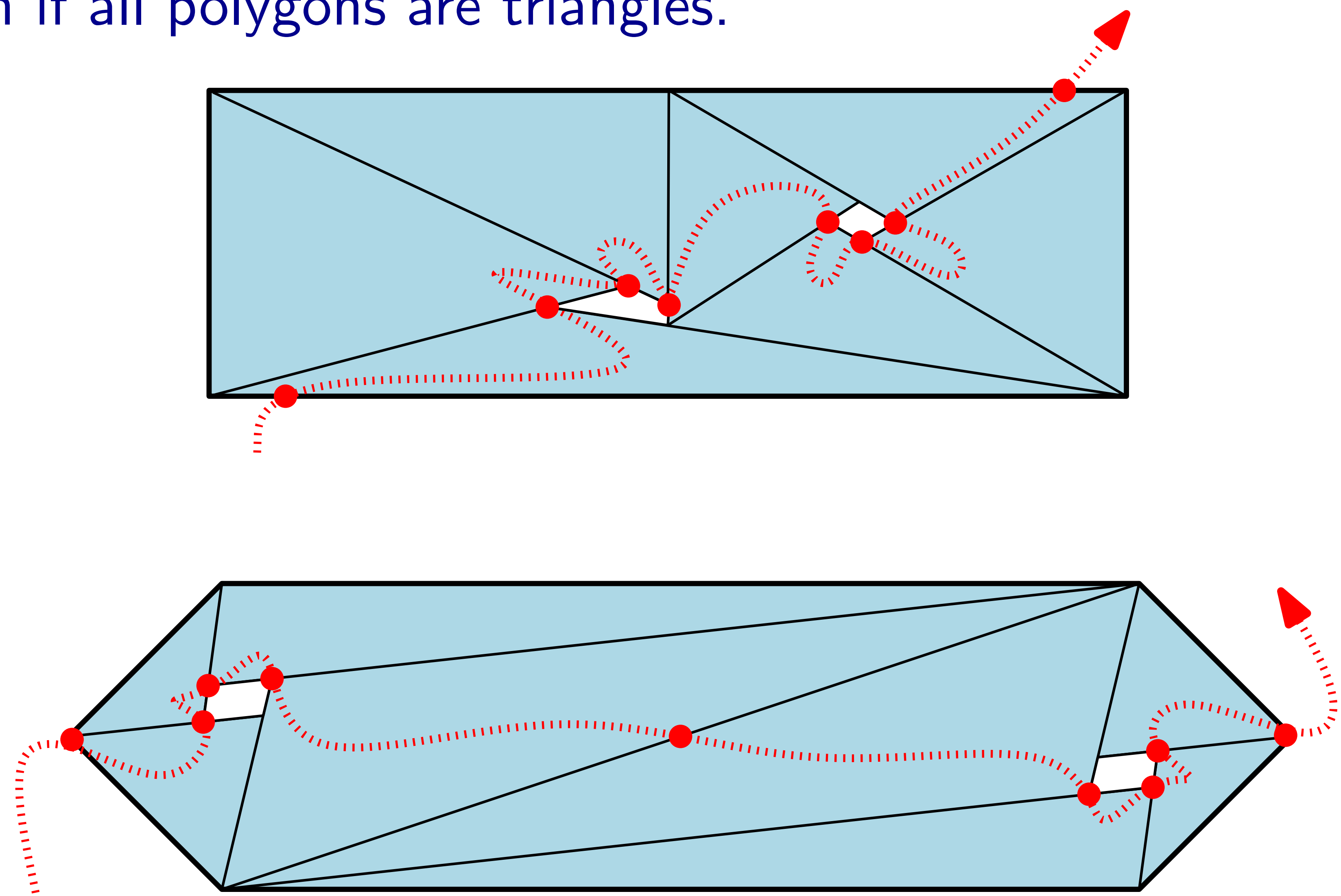
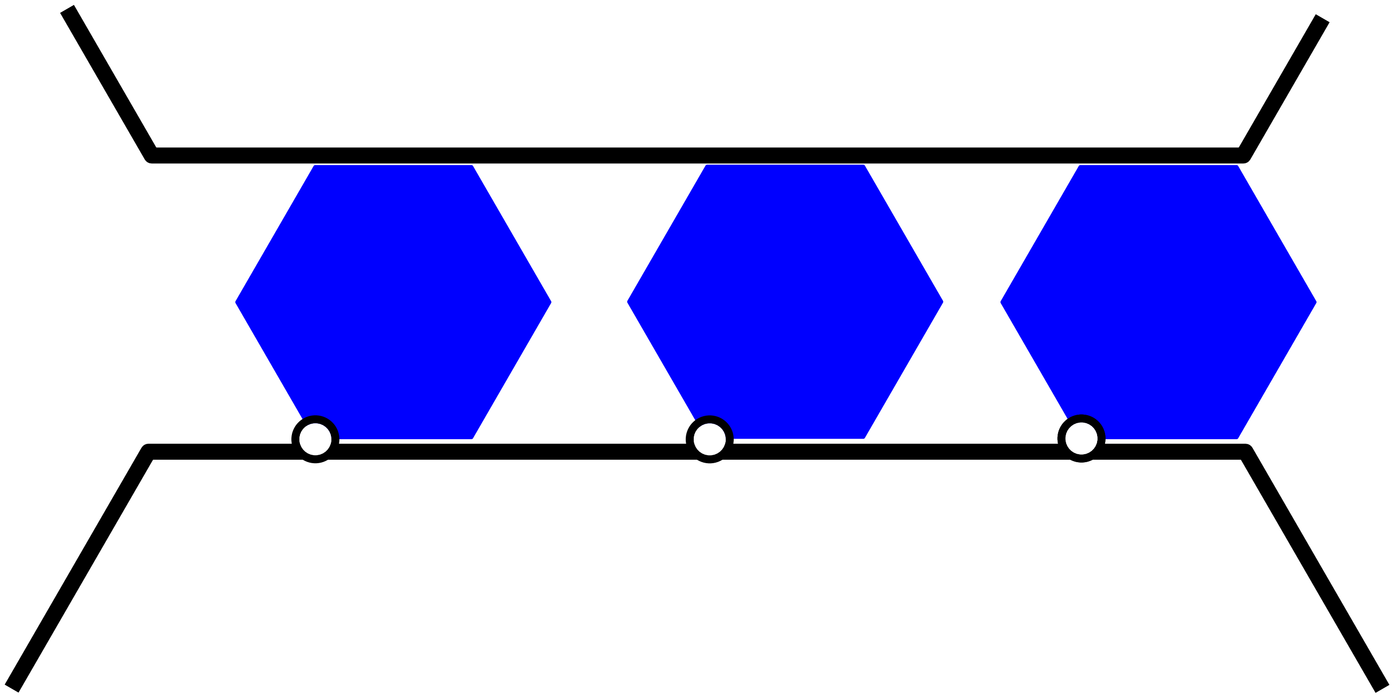


Body-Hinge Frameworks

For **trees**, both variants are strongly NP-hard (reductions from Planar3SAT and NAE3SAT).



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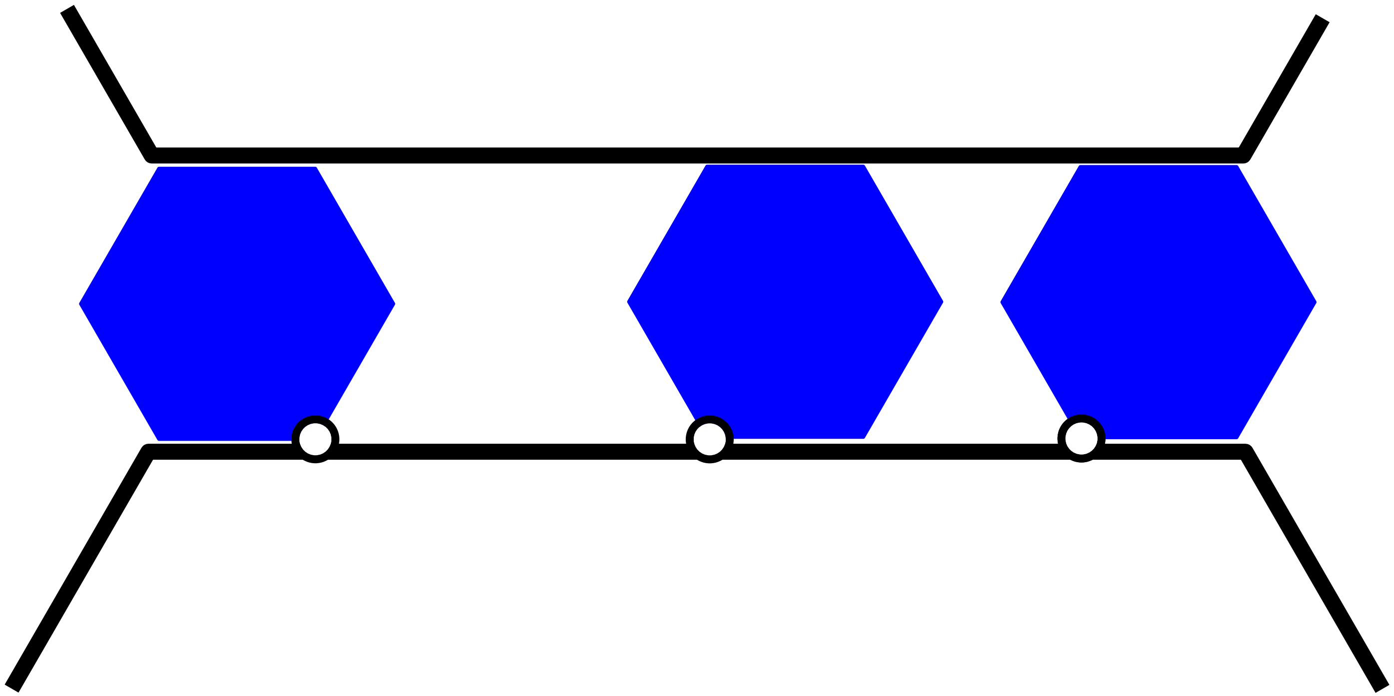

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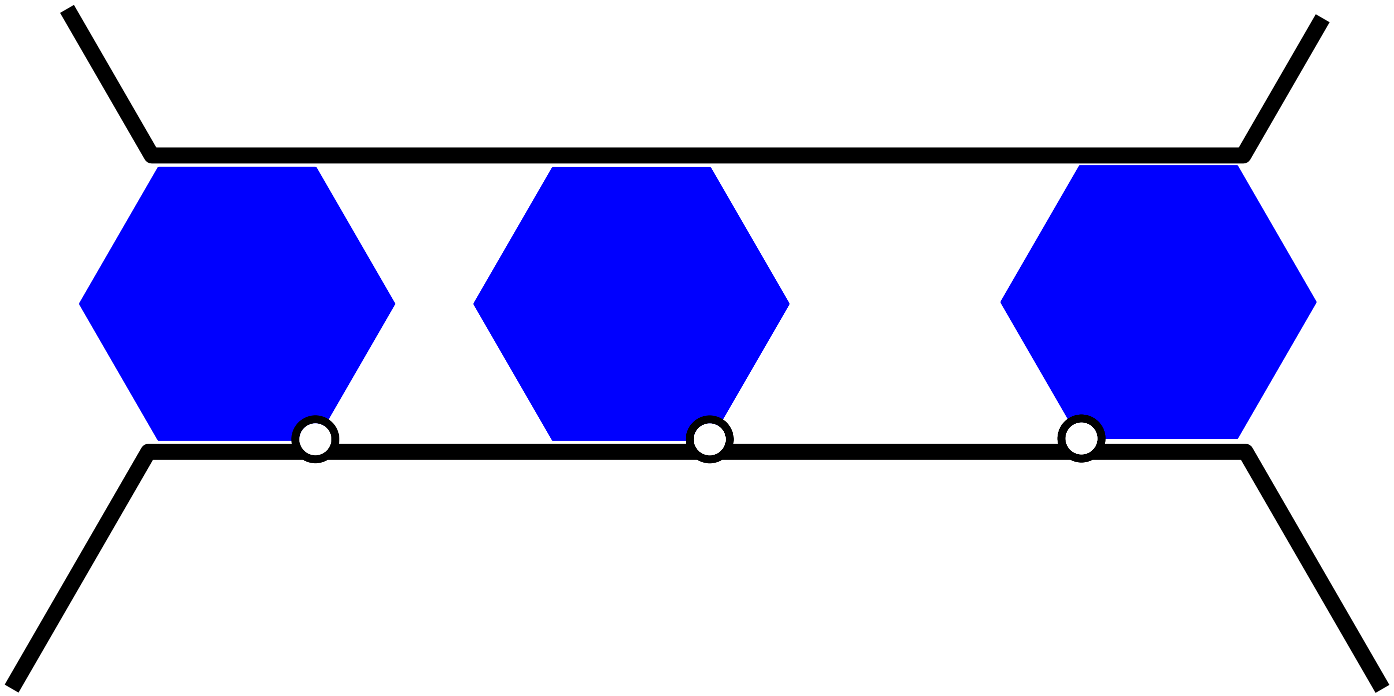

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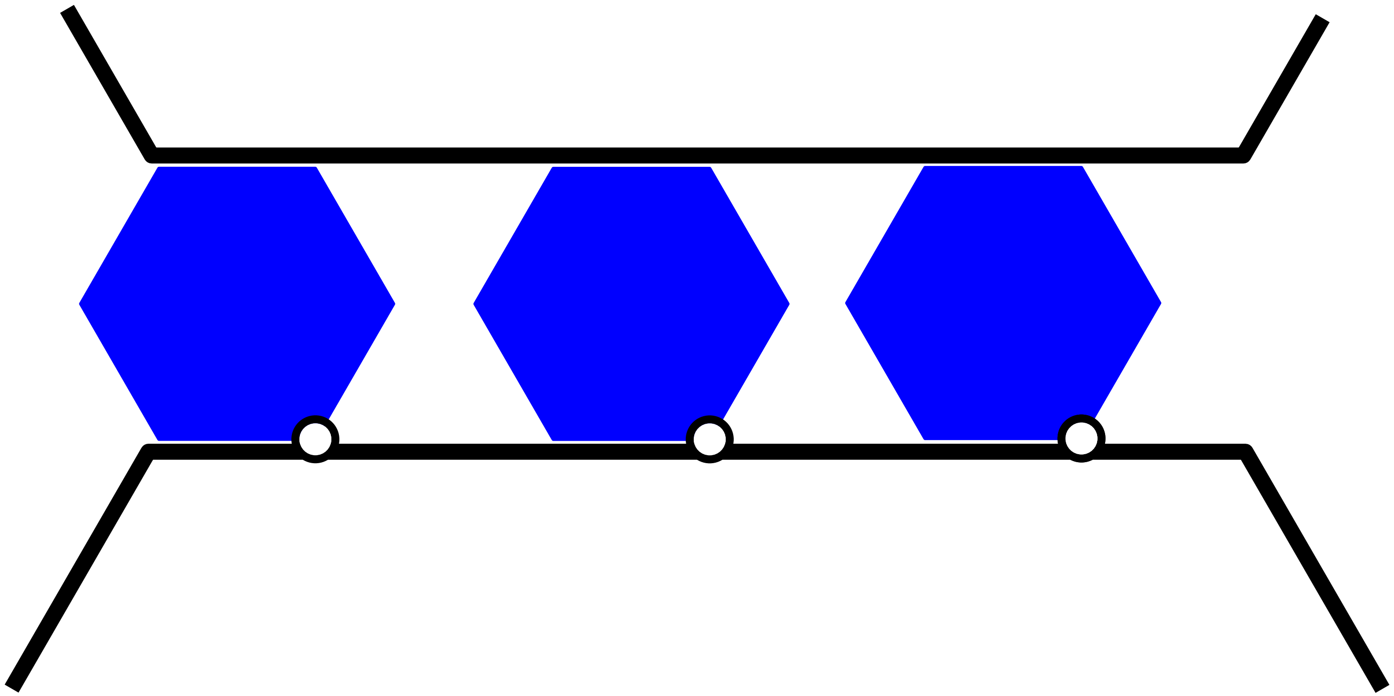

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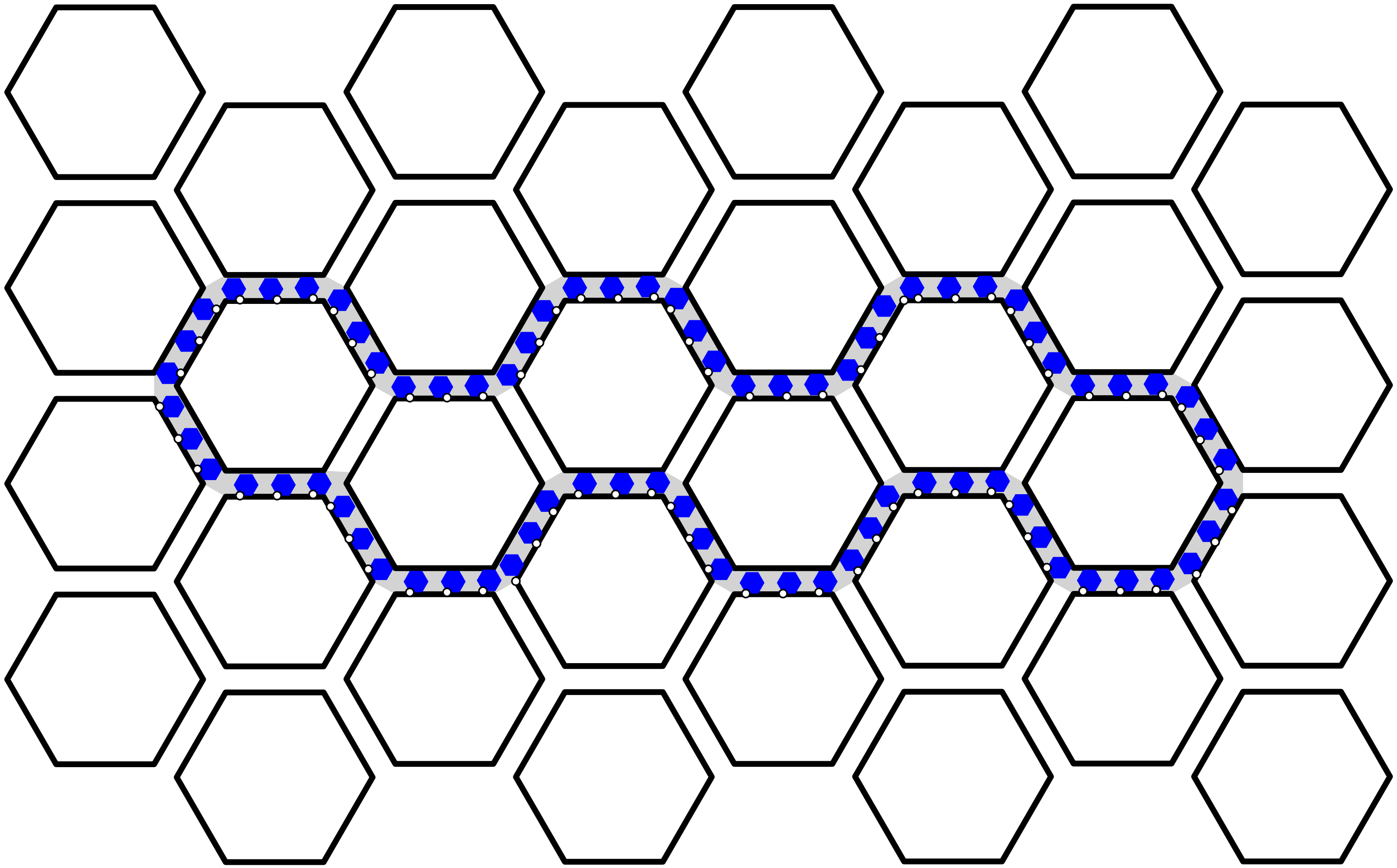

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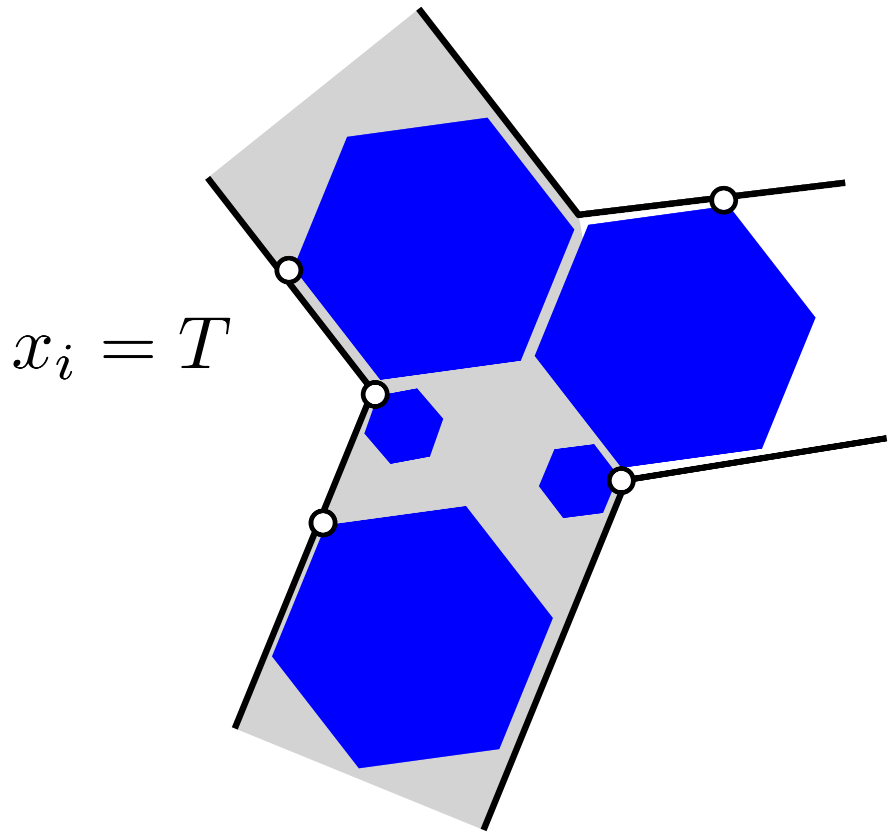

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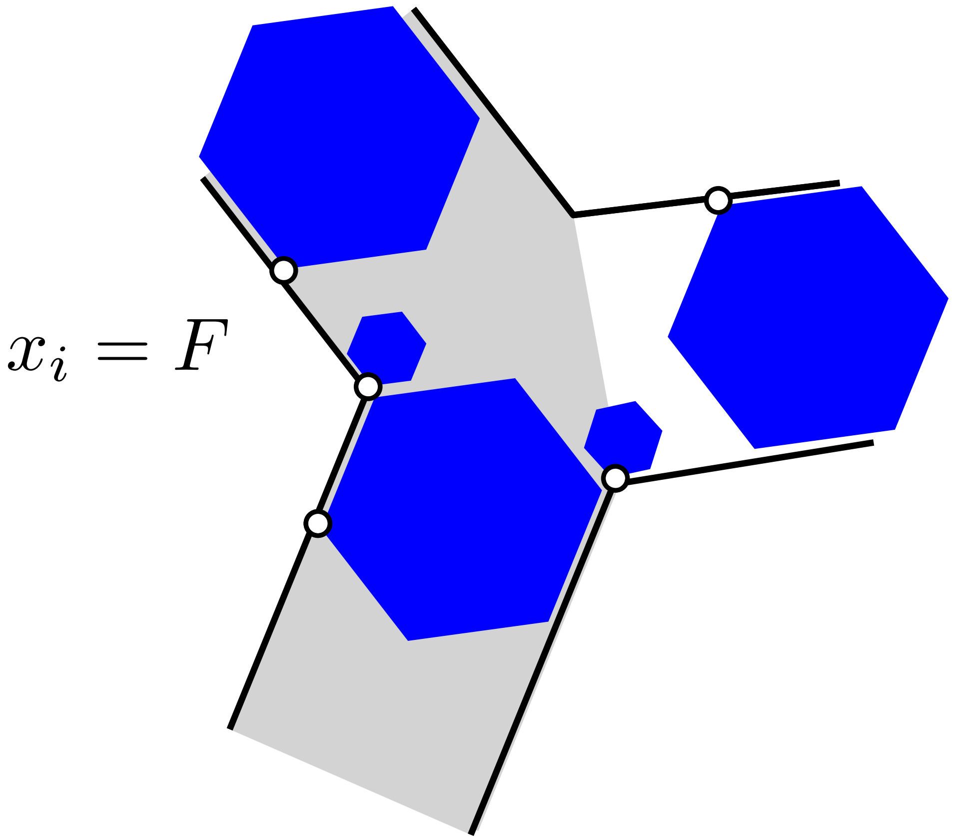

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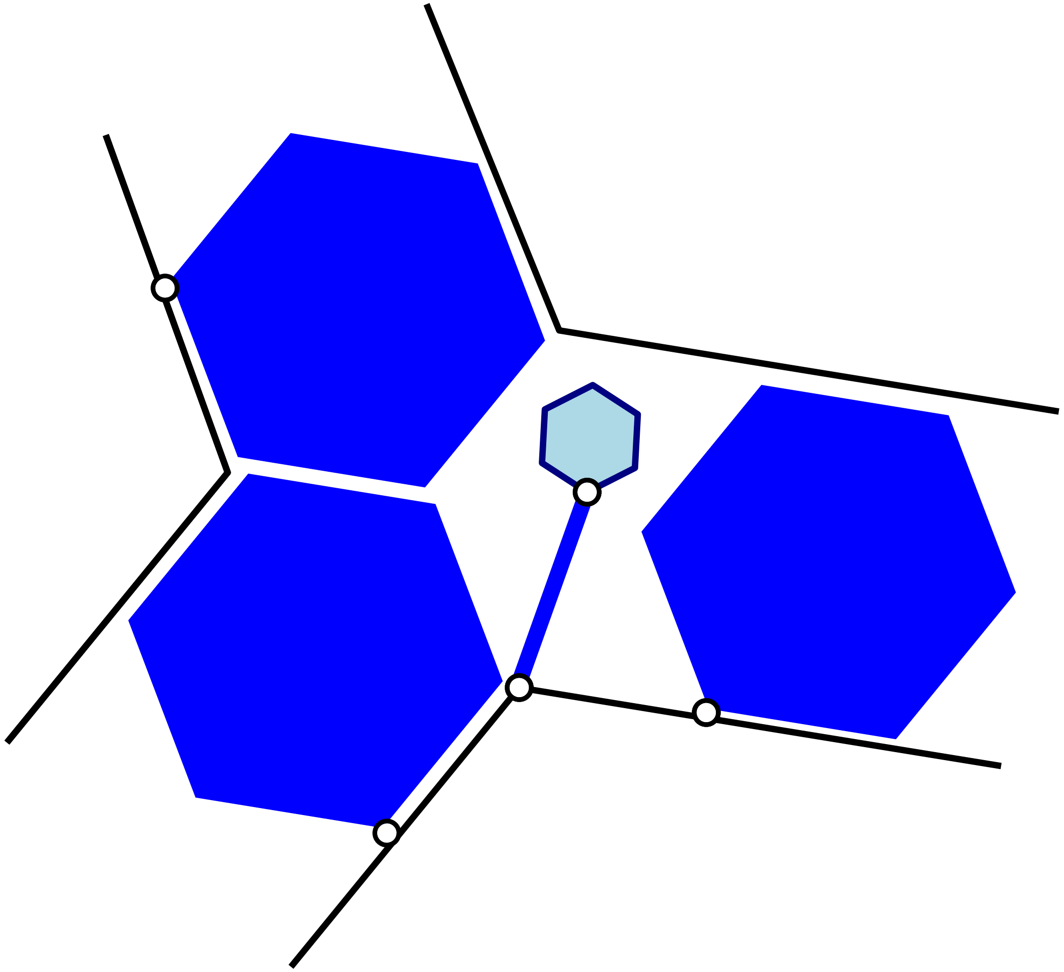

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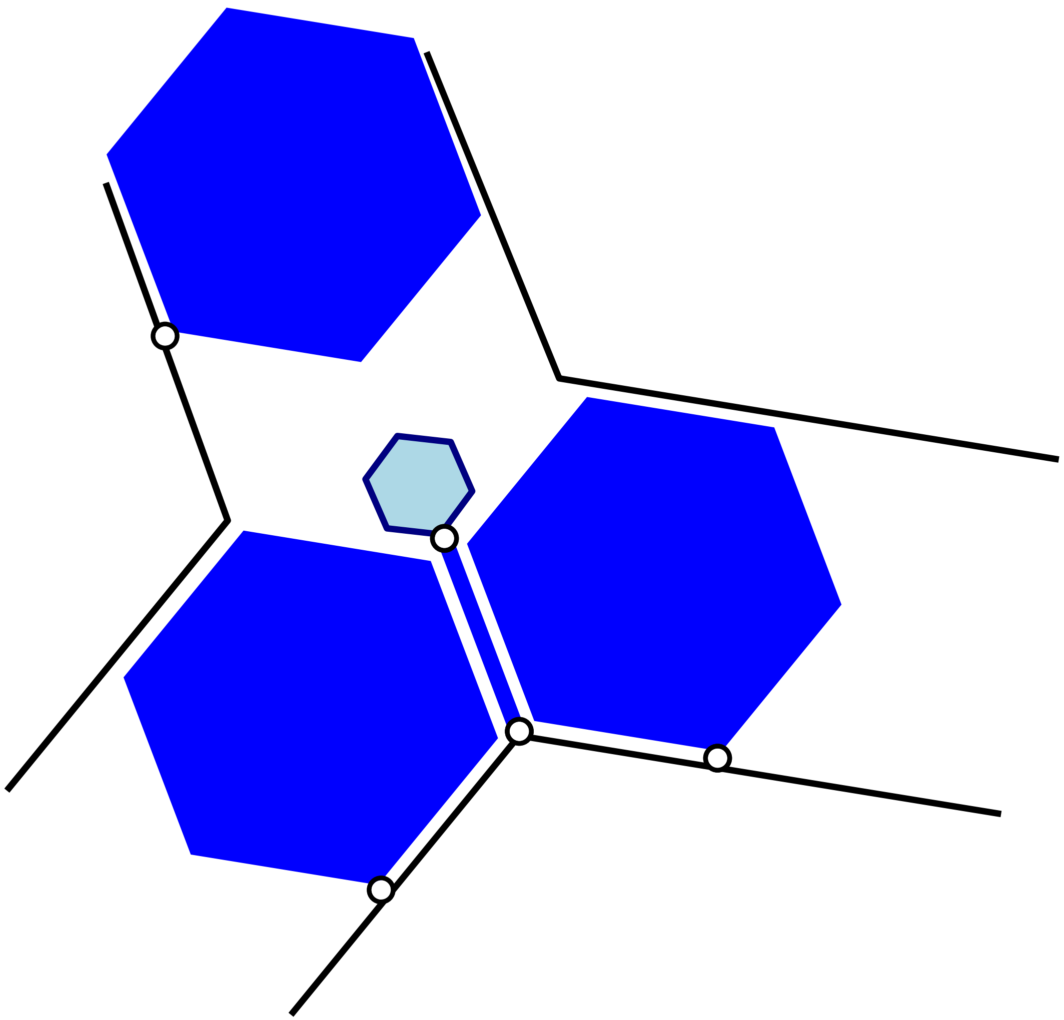

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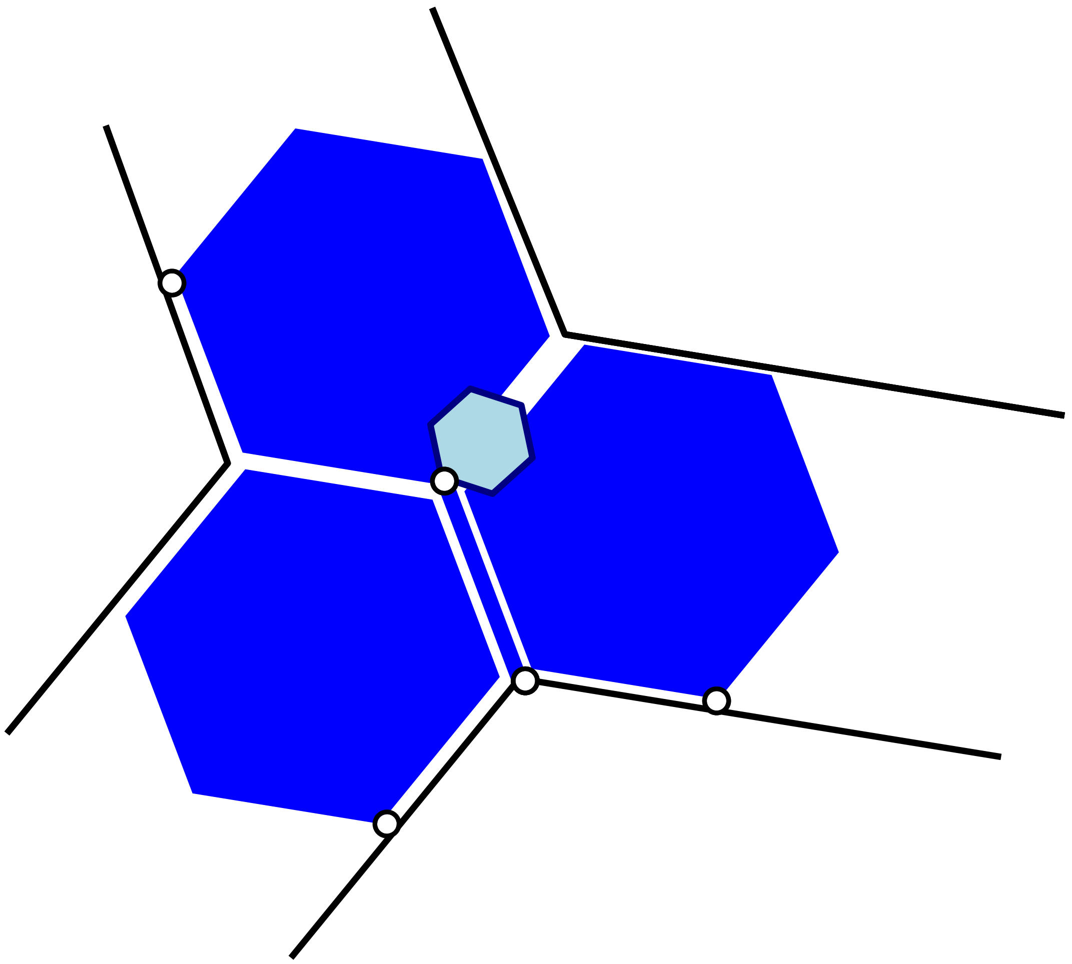
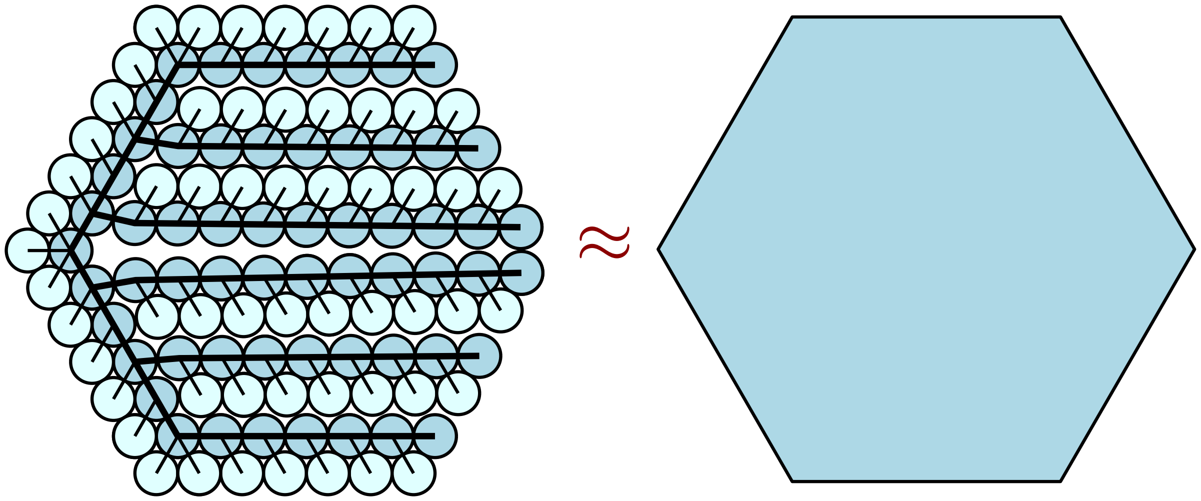


Body-Hinge Frameworks

Corollary: It is NP-hard to decide whether a **plane tree** is realizabile as a unit disk contact graph.



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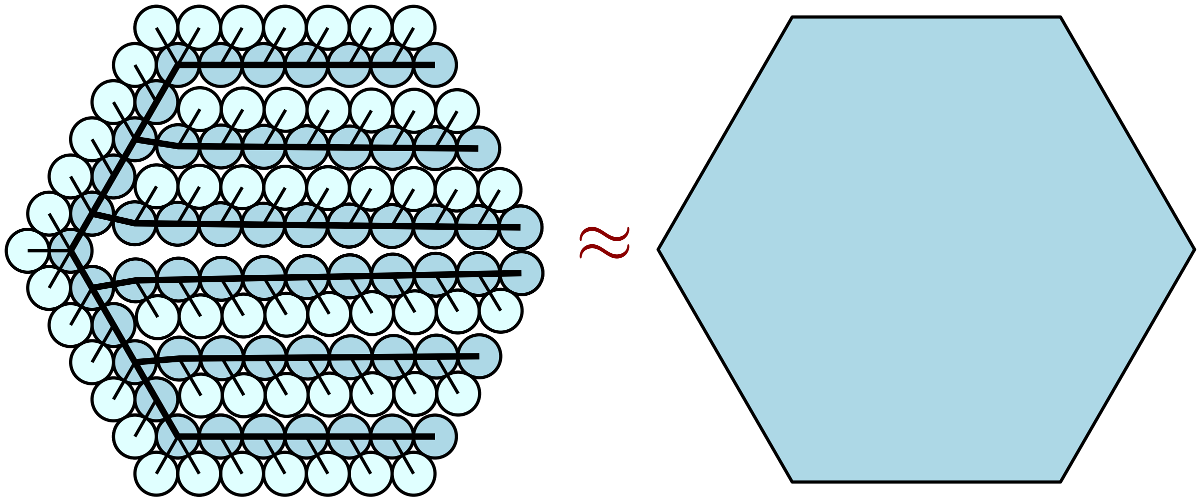
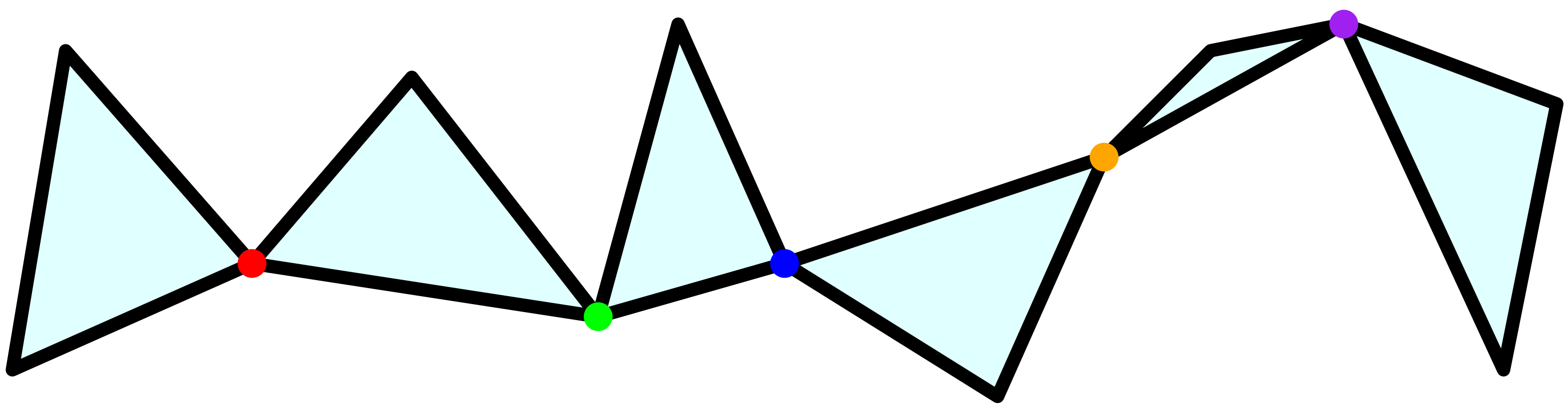


Observation: A path of triangles, hinged at distinct vertices, is realizable in $\mathbb{R}^2$.



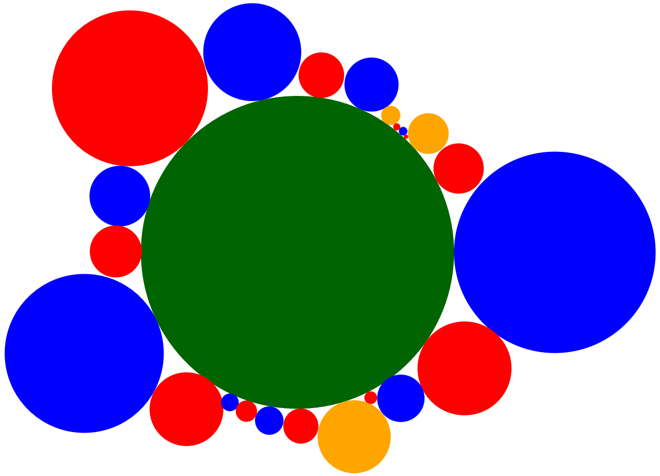
Body-Hinge Frameworks

Question: What is the complexity of recognizing contact **trees** of unit disks?

Question: What is the complexity of recognizing contact **binary trees** of unit disks?

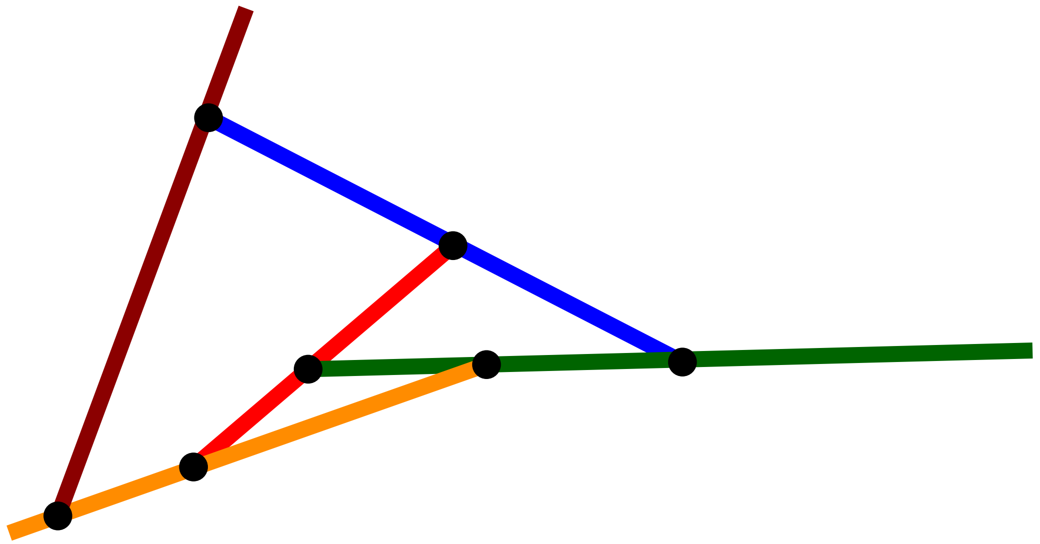
Klemz et al. (2015): For **stars**, it is NP-had to decide realizability as a contact graph of disks of *given radii*.

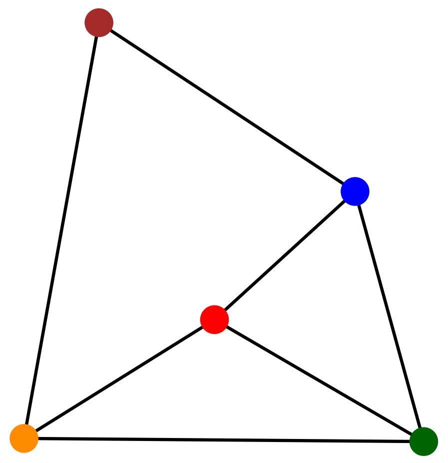
Problem: Every tree is realizable as a contact graph of disks of arbitrary radii. Find the smallest ratio between the maximum and minimum radii.



Contact Graph of Noncrossing Segments in $\mathbb{R}^2$

<u>no vertex-to-vertex contact</u>





Question: Can every arrangement of n noncrossing unit segments be continuously deformed into a *flat state* while maintaining the contact greaph?

Löffler & Tóth (2014): The answer is negative when the segments have arbitrary length.

Question: Is there an infinite family of contact graphs of unit segments that have a unique (or approximately unique) realization?

Thank you for your attention!