Geometric Representations of Graphs and the Existential Theory of the Reals

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Joint SoCG/STOC Workshops - June 18, 2016

The Existential Theory of the Reals

$$
\begin{aligned}
\exists x_1 \exists x_2 \exists x_3 \qquad & (x_1^2 + x_2^2 \leq 1) \\
\wedge \quad & (x_1 + x_3^3 = 0 \vee x_2^3 - x_1 > 0) \\
\wedge \quad & (x_1 + x_2 + x_3^3 = 1)
\end{aligned}
$$

The Existential Theory of the Reals

Sentences of the form

$$\exists x_1 \exists x_2 \ldots \exists x_n : F(x_1, x_2, \ldots, x_n)$$

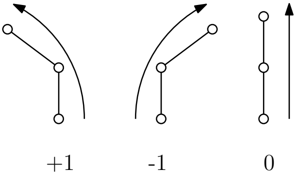
where F is a **Boolean combination of equalities and inequalities of polynomials** in the variables x_i.

- F describes a **semialgebraic set**.
- Computational problem: validity of the sentence over the reals
- $\exists\mathbb{R}$ is the class of computational decision problems that can be reduced in polynomial time to this problem.
- Robust to changes in the definition — Schaefer and Štefankovič, 2011
- $NP \subseteq \exists\mathbb{R} \subseteq PSPACE$ — Canny, 1988

$\exists\mathbb{R}$-complete Problems

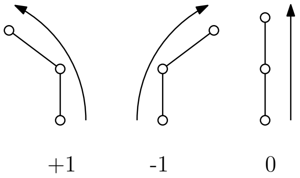
- A problem is $\exists\mathbb{R}$ is **$\exists\mathbb{R}$-complete** whenever every problem in $\exists\mathbb{R}$ can be reduced to it in polynomial time.
- Many **natural problems in computational geometry** can be shown to be $\exists\mathbb{R}$-complete.

Order Type of a Point Set



$$\operatorname{sign}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

Chirotopes



1. Cyclic symmetry: $\chi(p, q, r) = \chi(q, r, p)$,
2. Antisymmetry: $\chi(p, q, r) = -\chi(p, r, q)$,
3. **Grassmann-Plücker relations**: the set $\{\chi(p, q, r) \cdot \chi(p, s, t), -\chi(p, q, s) \cdot \chi(p, r, t), \chi(p, q, t) \cdot \chi(p, r, s)\}$ is either $\{0\}$ or contains both -1 and $+1$.
4. Identify χ with $-\chi$, and require that χ is not 0 everywhere.

One axiomatization of **rank-3 oriented matroids**

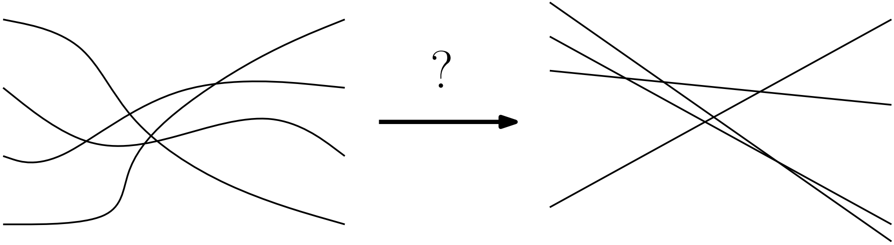
Realizability question

Given such a chirotope, does there exist a point set in $\mathbb{R}^2$ realizing it?

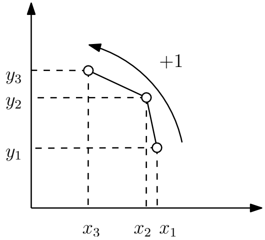
Topological Representation

- Folkman and Lawrence **Topological Representation Theorem** for oriented matroids (1974)
- Rank-3 oriented matroids have a topological representation as **arrangements of pseudolines** Bokowski, Mock, and Streinu, 2001
- **realizable** chirotope $\Leftrightarrow$ arrangement of **straight lines**

Stretchability



Realization Space



$$\{(x_1, y_1, x_2, y_2, x_3, y_3) : \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} > 0\} \subseteq \mathbb{R}^6$$

- The **realization space** of a rank-3 oriented matroid on n points is the subset of $\mathbb{R}^{2n}$, every point of which is a realization of the oriented matroid.
- The realization space of a rank-3 oriented matroid is a **semialgebraic set**.

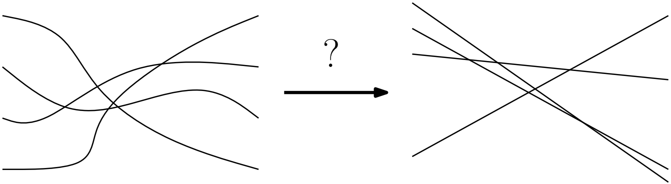
Mnëv's Theorem (∼1985)

Every semialgebraic set in $\mathbb{R}^d$ is **stably equivalent** to the realization space of some rank-3 oriented matroid.

Mnëv's Theorem ($\sim$1985)

Every semialgebraic set in $\mathbb{R}^d$ is **homotopy equivalent** to the realization space of some rank-3 oriented matroid.

Complexity-theoretic Consequence



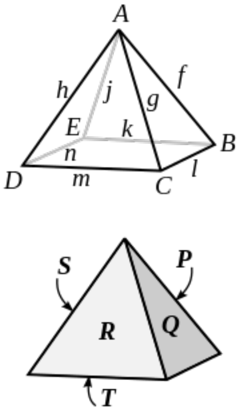
Deciding whether a pseudoline arrangement is stretchable is $\exists\mathbb{R}$-complete.

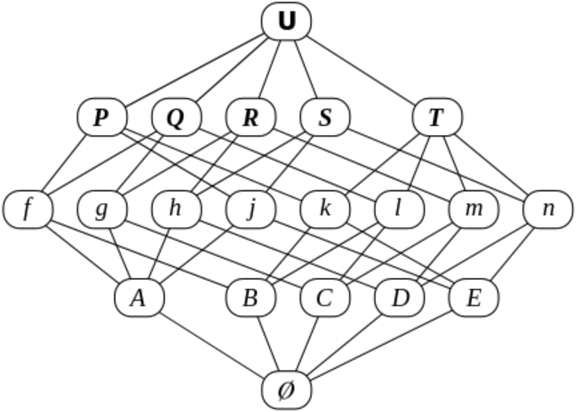
Complexity-theoretic Consequence

- A polynomial-time verifiable certificate for stretchability / chirotope realizability would yield $\exists\mathbb{R} \subseteq NP$.
- Realizations may require **exponentially many bits** – hence **doubly exponential coordinates** Goodman, Pollack, and Sturmfels, 1989

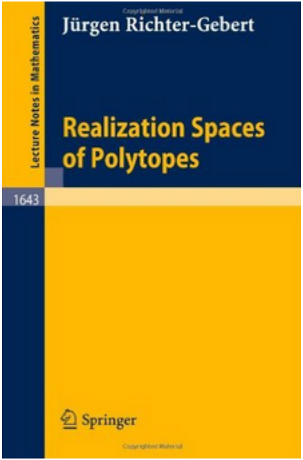
Some $\exists\mathbb{R}$-complete Problems

Polytope Realizability





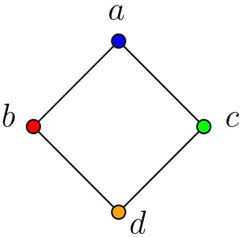
Polytope Realizability

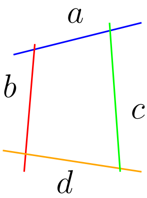


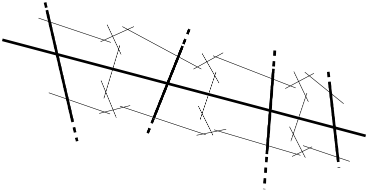
The realizability problem for **abstract 4-polytopes** is $\exists\mathbb{R}$-complete.

Richter-Gebert, 1996

Segment Intersection Graphs



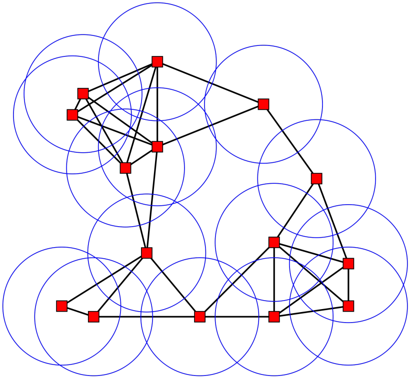




The recognition problem for segment intersection graphs is $\exists\mathbb{R}$-complete.

Kratochvíl and Matoušek, 1994

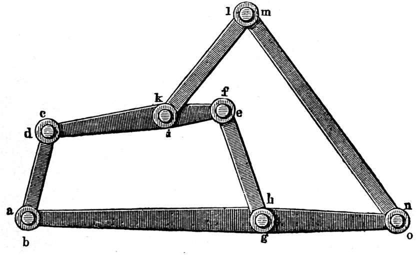
Disk Intersection Graphs

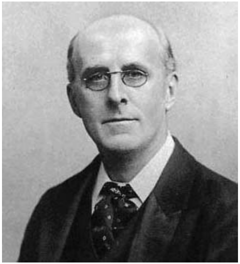


The recognition problem for unit disk graphs is $\exists\mathbb{R}$-complete.

McDiarmid and Müller, 2011

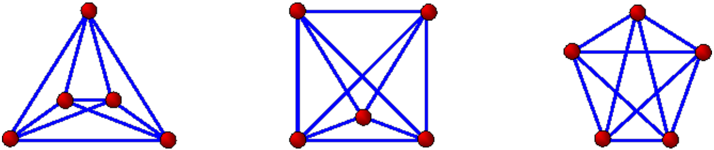
Linkages





Deciding whether a linkage is realizable in the plane is ∃ℝ-complete, even if all edges have unit length. Schaefer 2012

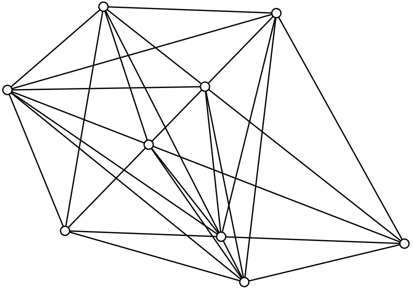
Rectilinear Crossing Number

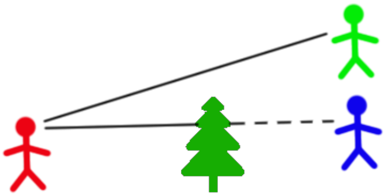


Given a graph G and an integer t, deciding whether the rectilinear crossing number of G is at most t is $\exists\mathbb{R}$-complete.

Bienstock 1991, and Schaefer 2009

Point Visibility Graphs

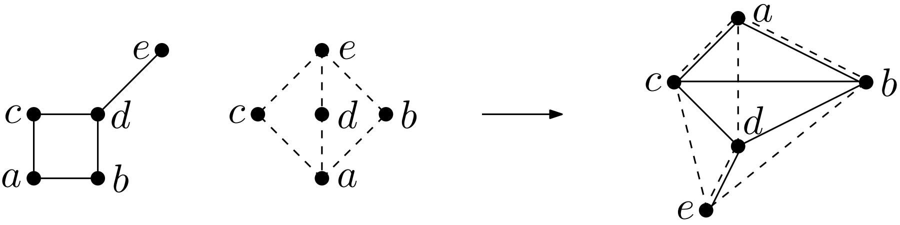




The recognition problem for point visibility graphs is $\exists\mathbb{R}$-complete.

C. and Hoffmann, 2015

Simultaneous Graph Embedding



The simultaneous graph embedding problem is $\exists\mathbb{R}$-complete. Kyncl, 2011

Computational Geometry Column 62.
ACM SIGACT News 46(4), December 2015.

Thank You!