Hanani–Tutte for radial drawings

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(Marcus Schaefer, De Paul Chicago and Michael Pelsmajer, IIT Chicago)

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Verified for the projective plane by *Pelsmajer et al.* (2009), and, ... hopefully, torus.

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Weak Hanani–Tutte theorem for monotone drawings

Pach & Tóth (2004): If we can draw a graph G in the plane such that
(i) every pair of edges cross evenly; and
(ii) projection $x(.)$ of every edge to x-axis is injective
then we can embed G such that (ii) still holds; $x(v)$ is unchanged for every vertex and the order of the end pieces of the edges at the vertices is unchanged.

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F., Pelsmajer, Schaefer & Štefankovič (2011): If we can draw a graph G in the plane such that
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Chimani et al. (2013) Our algorithm performs well in practice.

Radial Drawings

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Given a graph $G = (V, E)$ whose vertices are totally ordered a radial drawing of G is a drawing on the cylinder $\mathcal{C}$ such that
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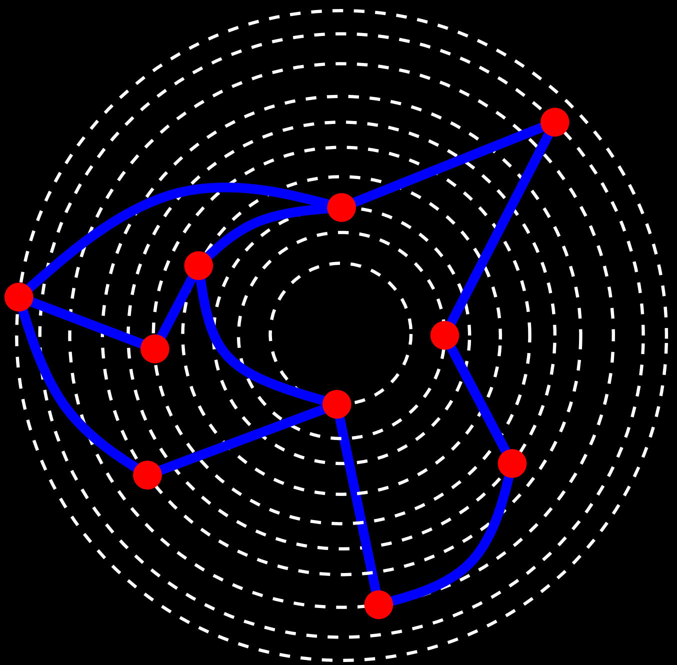

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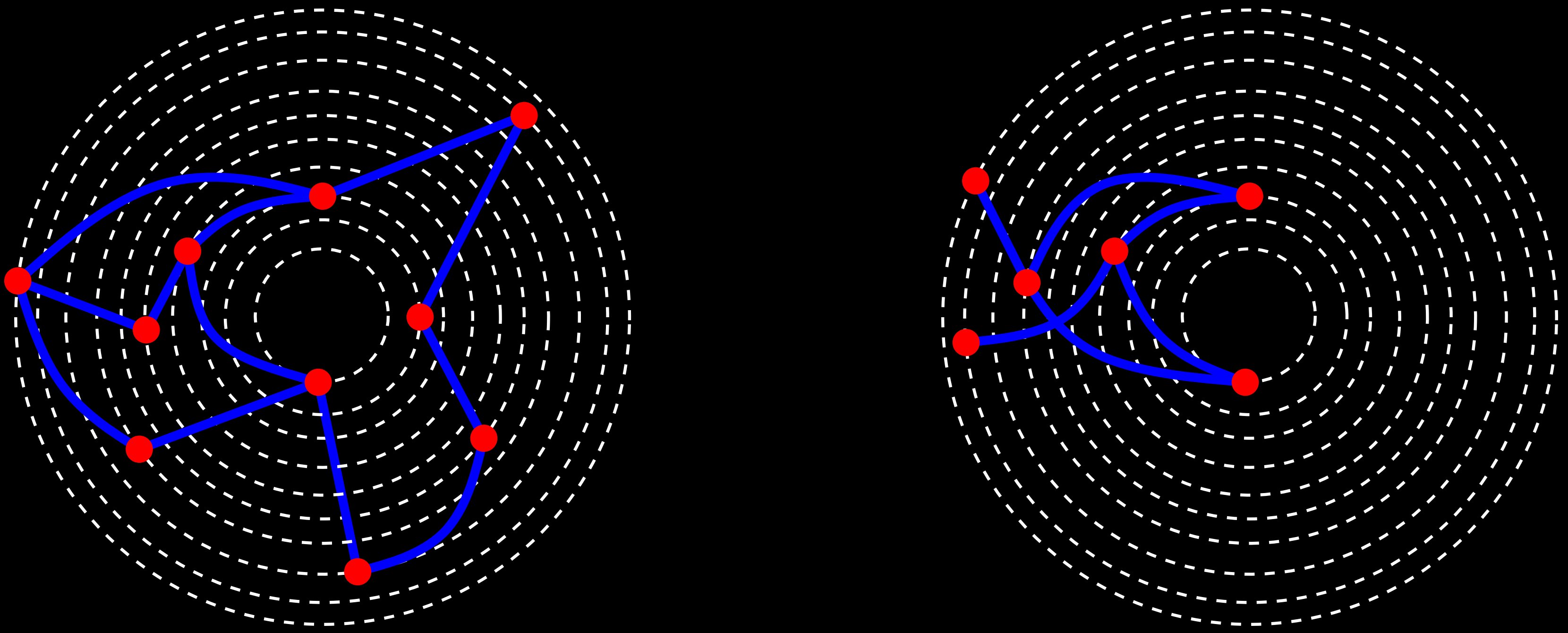

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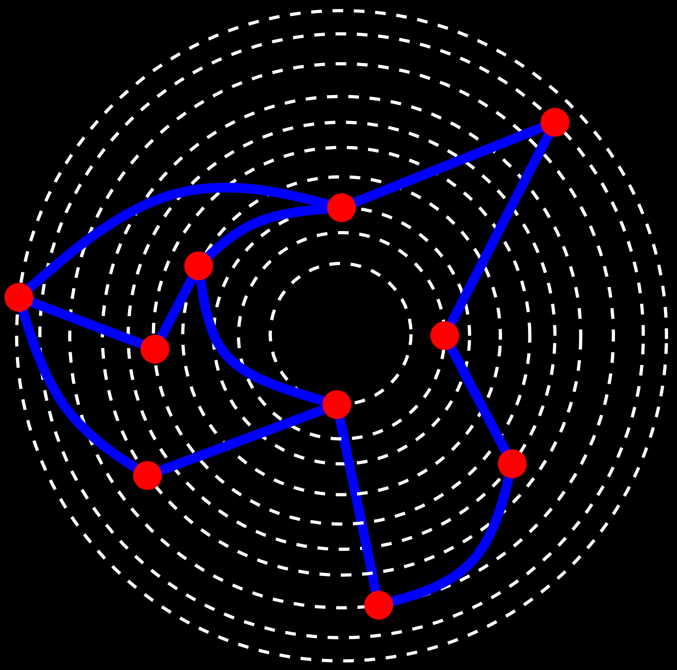

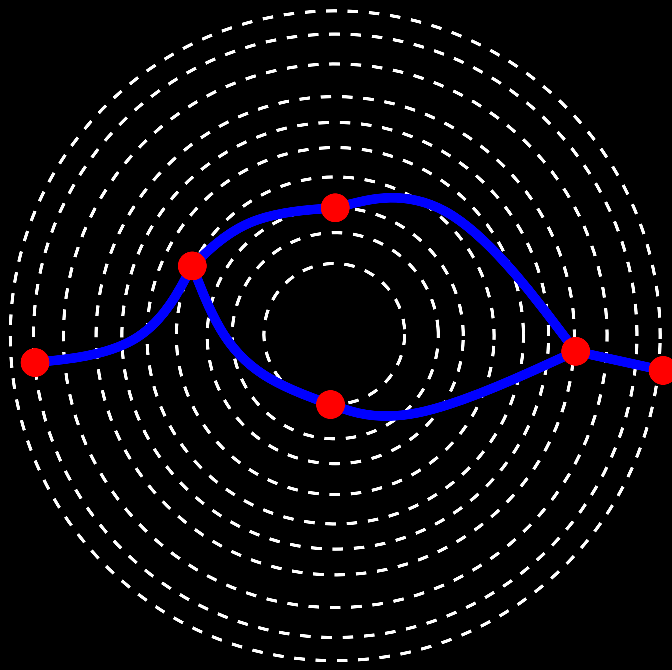

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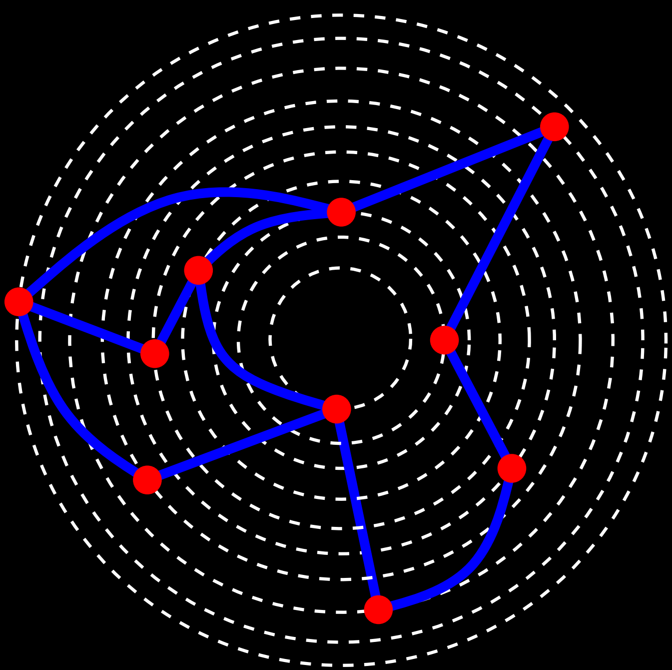

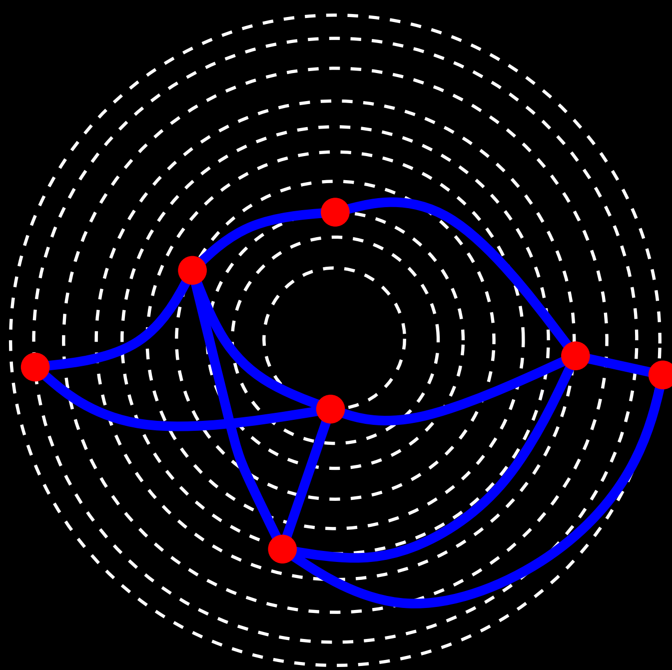

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Reduce ≤ 2-separations; *the weak variant* in the base case.

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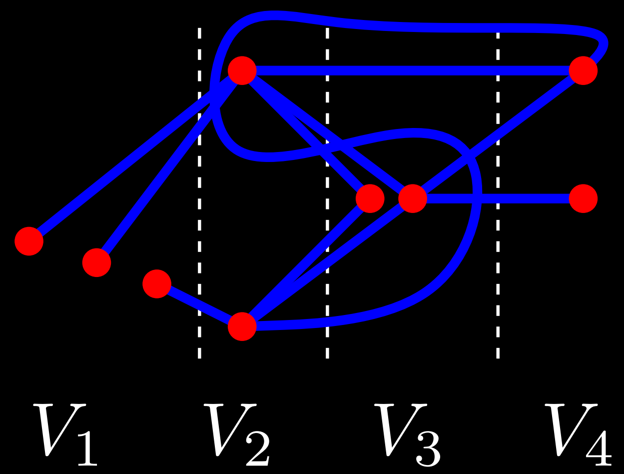

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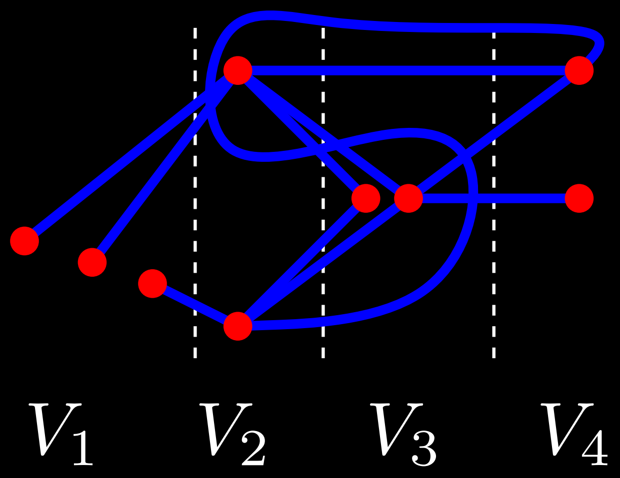

F. (2014) A graph $G = (V, E)$ equipped with a function $\gamma : V \to \mathbb{R}$ admits an x-bounded straight-line embedding if it admits an x-bounded drawing in which every pair of edges cross evenly.

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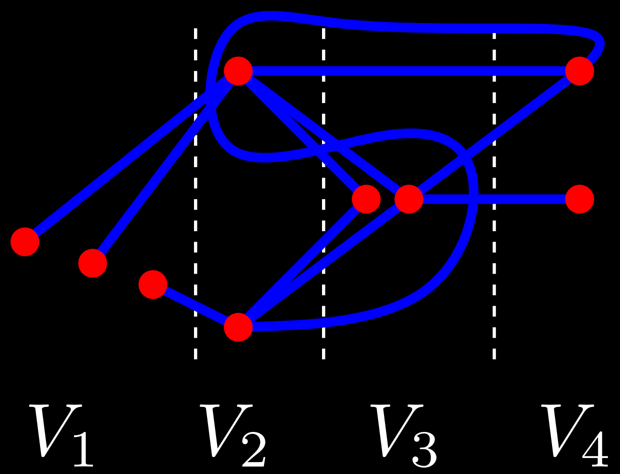

Conjecture. A graph $G = (V, E)$ equipped with a function $\gamma : V \to \mathbb{R}$ admits an x-bounded straight-line embedding if it admits an x-bounded drawing in which every pair of **non-adjacent** edges cross evenly.

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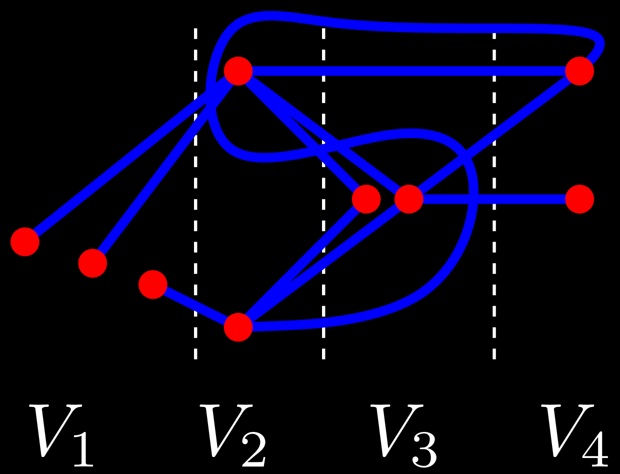

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F. (16+) The conjecture verified for graphs in which every connected component is either a tree or a generalized Θ-graphs.

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G is planar and $\mathcal{D}$ is the isotopy class of an embedding of G. $\mathcal{C} = (\mathcal{D}, \mathbb{Z}_2)$ is the corresponding chain complex: 2-dim. chains are generated by the inner faces of $\mathcal{D}$, 1-dim. chains by E, 0-dim. by V.

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Let $i_{\mathcal{D}}(C_1, C_2)$ denote the algebraic intersection number of the supports, which are balls, of pure chains C_1 and C_2 in $\mathcal{D}$ such that $dim(C_1) + dim(C_2) = 2$.

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F. (2016) $\mathcal{D}$ contains a straight-line x-bounded embedding iff $i_{\mathcal{D}}(C_1, C_2) = 0$ whenever $\gamma(C_1) \cap \gamma(\partial C_2) = \emptyset$ and $\gamma(\partial C_1) \cap \gamma(C_2) = \emptyset$, where $\gamma(.)$ is extended linearly to edges.

H-bounded Drawings

A drawing in the plane of a graph $G = (V, E)$ equipped with a graph homomorphism $\gamma : V \to H$, where $H = (V', E')$ is a plane graph, is (H, ϵ)-**bounded** if $\epsilon > \mathrm{dist}_{\mathcal{F}}(e, \gamma(e))$, for every $e \in E$.

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Let $i_{\mathcal{D}}(C_1, C_2)$ denote the algebraic intersection number of the supports, which are balls, of chains C_1 and C_2 in $\mathcal{D}$ such that $dim(C_1) + dim(C_2) = 2$.

Conjecture: $\mathcal{D}$ contains a straight-line (H, ϵ)-bounded embedding for every $\epsilon > 0$, iff (i) $i_{\mathcal{D}}(C_1, C_2) = i_H(\gamma(C_1), \gamma(C_2))$ whenever $\gamma(C_1) \cap \gamma(\partial C_2) = \emptyset$ and $\gamma(\partial C_1) \cap \gamma(C_2) = \emptyset$; and (ii) every face f of $\mathcal{D}$ admits an (H, ϵ)-bounded embedding $\mathcal{E}_f$ such that if $f \neq f'$ then $\mathcal{E}_f$ and $\mathcal{E}_{f'}$ do not "cover" the same face of H, i.e., winding numbers of faces of G w.r.t. to H are consistent.

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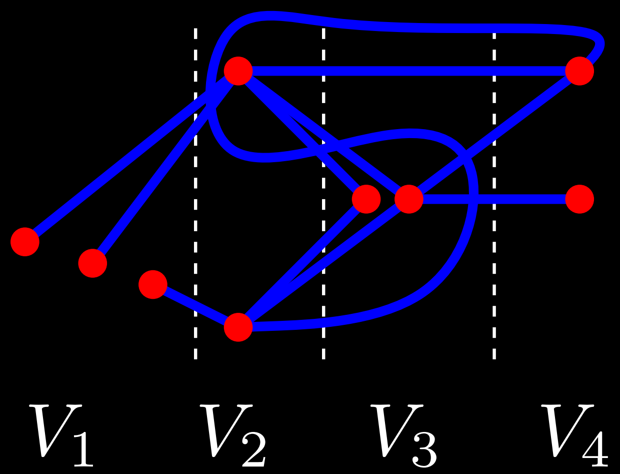

Conjecture. A graph $G = (V, E)$ equipped with a function $\gamma : V \to \mathbb{R}$ admits an x-bounded straight-line embedding if it admits an x-bounded drawing in which every pair of **non-adjacent** edges cross evenly.

Does at least the **weak** variant of the radial version of the conjecture hold?

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Given a graph $G = (V, \overrightarrow{E})$, whose vertices are cylically ordered, a cyclic drawing of G is a drawing on the cylinder $\mathcal{C}$ such that
(i) Values $\mathbb{S}^1(v),\ v \in V$, respect the given order; and
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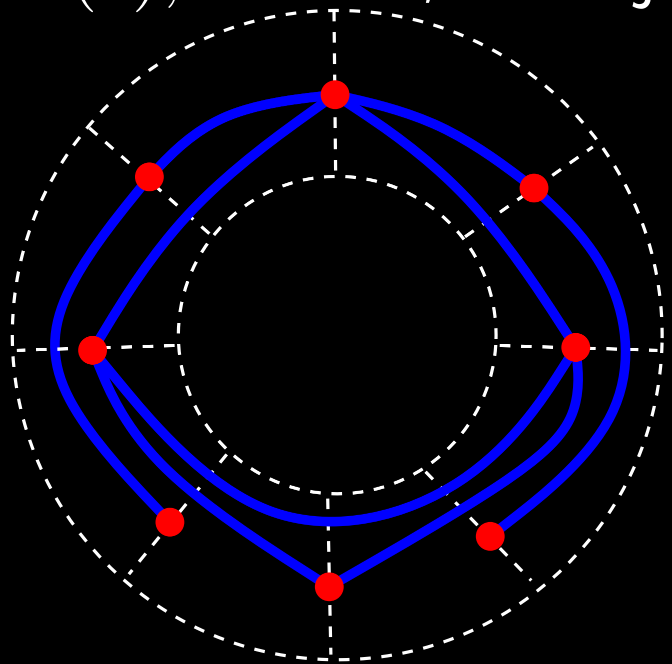

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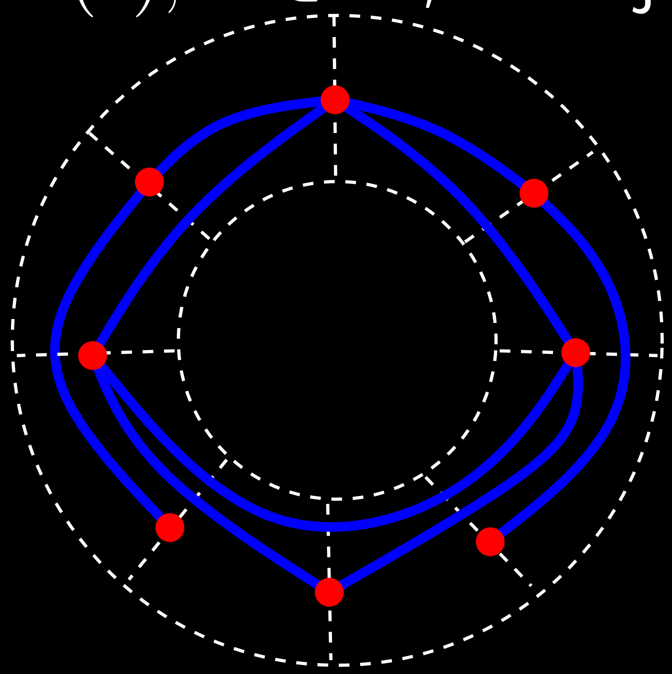
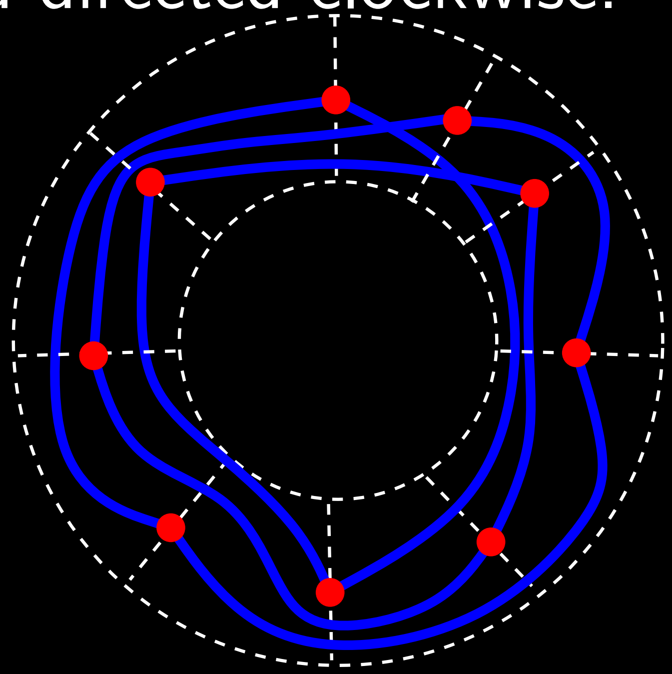
Given a graph $G = (V, \overrightarrow{E})$, whose vertices are cylically ordered, a cyclic drawing of G is a drawing on the cylinder $\mathcal{C}$ such that
(i) Values $\mathbb{S}^1(v),\ v \in V$, respect the given order; and
(ii) $\mathbb{S}^1(e),\ e \in \overrightarrow{E}$, are injective and directed clockwise.





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