Homework 1

Due to October 15, 23:59.

Let P be a poset and $x \in P$. Let $U(x)$ and $D(x)$ be the *open* up-set/down-set of x in P that is

$$U(x) = \{y > x \mid y \in P\},$$
$$D(x) = \{y < x \mid y \in P\}.$$

Exercise 1. Prove that for every interval order P, we have

$$|\{D(x) \mid x \in P\}| = |\{U(x) \mid x \in P\}|.$$

Exercise 2. A T_0-topology on a set S is a set-system $\Sigma \subseteq \mathrm{Pot}(S)$ with the properties

(i) $\emptyset, S \in \Sigma$.
(ii) When $A, B \in \Sigma$, then $A \cup B \in \Sigma$.
(iii) When $A, B \in \Sigma$, then $A \cap B \in \Sigma$.
(iv) For $a, b \in S$ there is an $A \in \Sigma$ with $a \in A$ and $b \notin A$ and/or there is a $B \in \Sigma$ with $b \in B$ and $a \notin B$. (weak separation axiom).

Show that T_0-topologies and partial orders on S are in bijection.

Exercise 3. Let $P = (X, \leqslant)$ be an order with a weighting $w : X \to \mathbb{R}_+$ on the elements. Show that there is a weighting $g : \mathcal{A} \to \mathbb{R}_+$ of the antichains of P such that $\max(w(C) : C$ chain $) = \sum_A g(A)$ and $w(x) = \sum_{A: x \in A} g(A)$ for all $x \in X$.
(Note that this is a weighted version of the dual of Dilworth's Theorem.)

Exercise 4. Find and read the Gallai-Milgram Theorem from at least two different sources. What is the connection of the statement with the course content?

Exercise 5. Show, that every infinite poset contains an infinite chain or an infinite antichain.