Homework 3

Due to October 29, 23:59.

Exercise 1. For each k with $1 \leqslant k \leqslant n/2$ find an intersecting family $\mathcal{F}_k$ of size 2^{n-1} in $\mathcal{B}_n$ such that the smallest set in $\mathcal{F}_k$ has size k.

Exercise 2. A family of subsets $\mathcal{F}$ of $[n]$ is *distinguishing* if for every $x \neq y \in [n]$ there is $F \in \mathcal{F}$ so that $|F \cap \{x, y\}| = 1$. A family of subsets $\mathcal{F}$ of $[n]$ is *strongly distinguishing* if for every $x \neq y \in [n]$ there are $F_1, F_2 \in \mathcal{F}$ such that $x \in F_1 - F_2$ i $y \in F_2 - F_1$.

(i) What is the minimum size of a distinguishable subset of $[n]$?
(ii) What is the minimum size of a strongly distinguishable subset of $[n]$?

Exercise 3. Let $\mathcal{F} = \{A_1, \ldots, A_m\}$ be a family of sets each of size r and let $\mathcal{G} = \{B_1, \ldots, B_m\}$ be a family of sets each of size s such that

(i) $A_i \cap B_i = \emptyset$ for each $i \in [m]$,
(ii) $A_i \cap B_j \neq \emptyset$ for all $i \neq j$, $i, j \in [m]$.

Show that

$$m \leqslant \binom{r+s}{s}.$$

Hint. Show first that

$$\sum_{i=1}^{m} \frac{1}{\binom{|A_i|+|B_i|}{|A_i|}} \leqslant 1.$$

Exercise 4. Show that there cannot be a skew version of the inequality from the previous exercise. More specifically, for every integer n, find sequences of finite sets $A_1, \ldots, A_m$ and $B_1, \ldots, B_m$ such that

(i) $A_i \cap B_i = \emptyset$ for each $i \in [m]$,
(ii) $A_i \cap B_j \neq \emptyset$ for all $i > j$, $i, j \in [m]$, but

$$\sum_{i=1}^{m} \frac{1}{\binom{|A_i|+|B_i|}{|A_i|}} \geqslant n + 1.$$

Exercise 5. Let σ be a cyclic permutation of $[n]$ and let $\mathcal{A}$ be a family of k-arcs such that for every h tuple $(A_1, A_2, \ldots, A_h)$ of sets from $\mathcal{A}$ we have $A_1 \cap A_2 \cap \ldots \cap A_h \neq \emptyset$. Show that if $k \cdot h \leqslant (h-1)n$, then $|\mathcal{A}| \leqslant k$.