Homework 4

Due to November 5, 23:59.

Exercise 1. Let n, k be integers with $n, k \geqslant 2$. The *generalized shift graph* $G_n^{(k)}$ is the graph with the vertex set $V(G_n^{(k)}) = \binom{[n]}{k}$ and $(x_1 < \cdots < x_k)$, $(x_2 < \cdots < x_{k+1})$ are adjacent for every $x_1, \ldots, x_{k+1}$ with $1 \leqslant x_1 < \cdots < x_k < x_{k+1} \leqslant n$.

(i) What is the minimum length of a cycle in $G_n^{(k)}$?
(ii) What is the minimum length of an odd cycle in $G_n^{(k)}$?

Exercise 2. Let $\mathcal{F} = \{A_1, \ldots, A_m\}$ be a family of sets each of size r and let $\mathcal{G} = \{B_1, \ldots, B_m\}$ be a family of sets each of size s such that

(i) $A_i \cap B_i = \emptyset$ for each $i \in [m]$,
(ii) $A_i \cap B_j \neq \emptyset$ for all $i < j$, $i, j \in [m]$.

Show that

$$m \leqslant \binom{r+s}{s}.$$

Exercise 3. The *local chromatic number* of a graph G is

$$\Psi(G) := \min_{c} \max_{v \in V(G)} |\{c(u) \mid u \in N(v)\}| + 1,$$

where the minimum is taken over all proper vertex-colorings c of G.

Show that $\Psi\left(G_n^{(2)}\right)$ goes to infinity when n grows.

Exercise 4. We have seen that posets with the same cover graph need not have the same dimension. How large can the difference be?

Exercise 5. For all integers $g, d \geqslant 1$, there exists a poset with a comparability graph of girth at least g and dimension at least d.