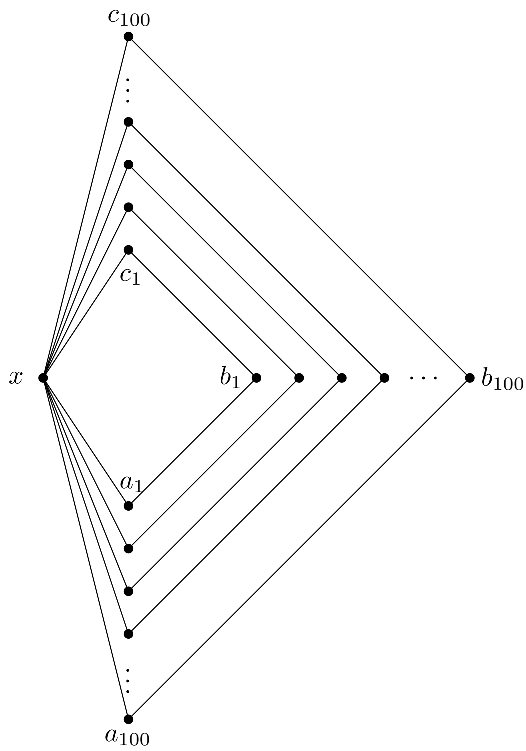
Homework 5

Due to November 12, 23:59.

Exercise 1. The poset depicted below has a planar diagram and an outerplanar cover graph. What is the dimension of this poset?



Exercise 2. Read and digest in-depth the proof of Frankl's Theorem given in *Thirty-three Miniatures: Mathematical and Algorithmic Applications of Linear Algebra* by Jiřì Matoušek. You can find it here: [pdf, Miniature 33]. Write the amount of time you spent on this proof. Is there something you like/dislike about it?

A 3-*orientation* of a triangulation is an orientiation of inner edges such that each inner vertex has exactly 3 outgoing edges. The outer edges remain undirected.

cw - clockwise
ccw - counterclockwise

Exercise 3. Let $\mathcal{T}$ be a 3-orientation of a triangulation G. Show that

(i) If C is a directed chordless cycle in $\mathcal{T}$ such that for every edge e incident with exactly one vertex in C and contained in the region bounded by C, the edge e is oriented towards a vertex in C, then C is a triangle.

(ii) Prove that every 3-orientation $\mathcal{T}$ of a triangulation G gives a unique Schnyder wood (up to cyclic shift of colors RGB) such that the orientation of edges in $\mathcal{T}$ agrees with the orientation of edges in the Schnyder wood. Give an algorithm producing such a Schnyder wood.

Let G be a 4-connected triangulation and $\mathcal{T}$ be a 3-orientation of G. A face f of G is *flippable* if all edges bounding f are oriented ccw.

A *flip* of a flippable face f is a modification of the 3-orientation that changes orientations of all edges around f from ccw to cw. Such a new 3-orientation is denoted by $\mathcal{T}_f$.

$(f_1, f_2, \ldots)$ is a *flip-sequence* in $\mathcal{T}$ if f_1 is a flippable face in $\mathcal{T}$, f_2 is a flippable face in $\mathcal{T}_{f_1}$, f_3 is a flippable face in $\mathcal{T}_{f_1,f_2}$, and so on.

Exercise 4. Let G be a 4-connected triangulation with a 3-orientation $\mathcal{T}$.

(i) If $\mathcal{T}$ contains a cw (ccw) cycle C then it also contains a cw (ccw) triangle contained in the region bounded by C.
(ii) Let C be a ccw oriented cycle in $\mathcal{T}$. Prove that there is a flip-sequence S such that $\mathcal{T}_S$ is exactly like $\mathcal{T}$ except the edges of C which are cw in $\mathcal{T}_S$.
(iii) Prove that every flip-sequence S in $\mathcal{T}$ is finite.

Exercise 5.

(i) Let G be a triangulation and consider two distinct 3-orientations of G. Prove that there is a cycle C in G that is oriented cw in one 3-orientation and ccw in the other.
(ii) Let G be a 4-connected triangulation. Prove that there is exaclty one 3-orientation with all cycles oriented cw.