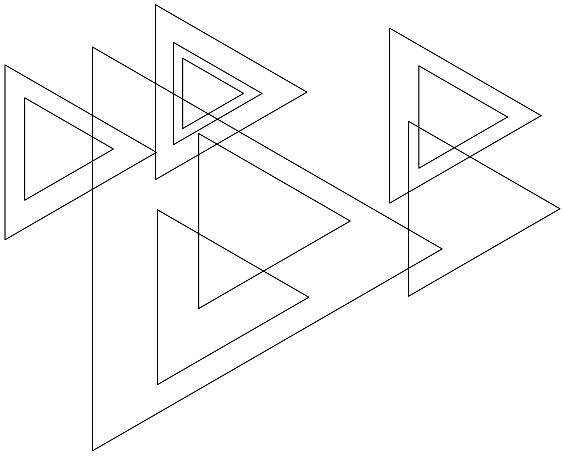
Homework 6

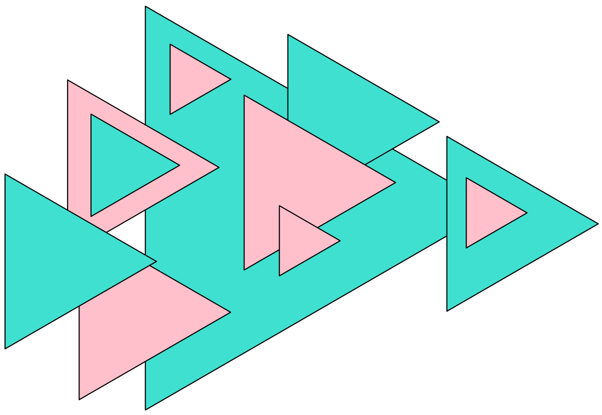
Due to November 19, 23:59.

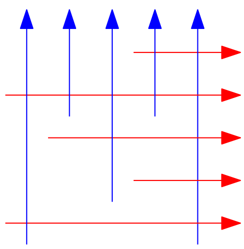
Exercise 1.

(i) What is the maximum dimension of a containment order of homothethic triangles in the plane?



(ii) Let G be a bipartite graph with vertices represented by two-colored homothethic triangles in the plane and two triangles of different colors are adjacent if and only if they intersect. (Note that triangles of the same color can intersect but still they are not adjacent.) What is the maximum possible dimension of G seen as a poset of height 2 with one color being minimas and the other maximas?



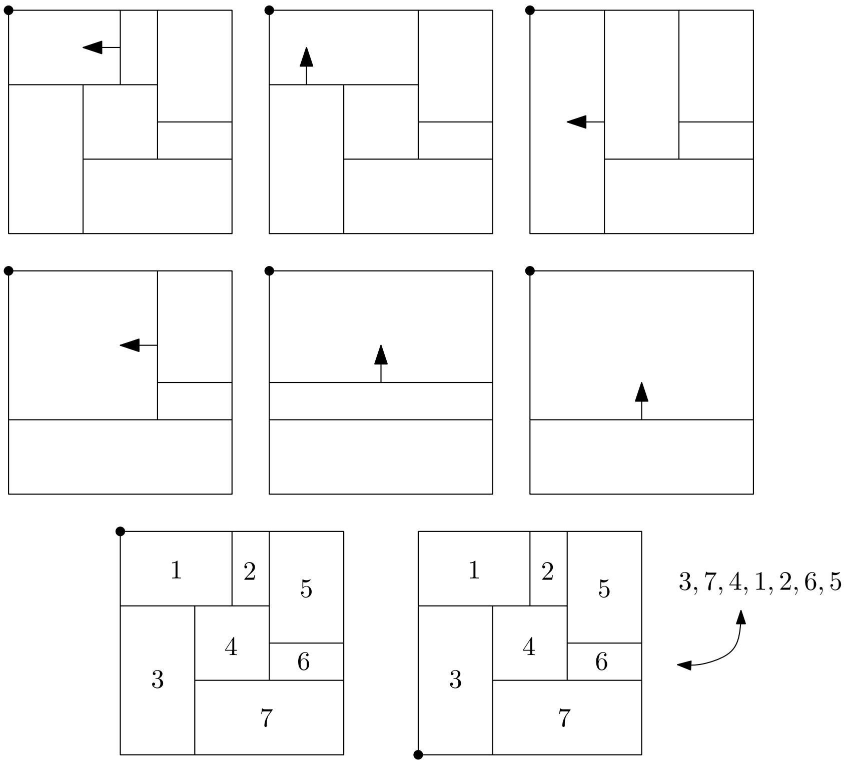


(iii) What is the maximum dimension of an intersection graph (seen as a height 2 poset again) of horizontal rays directed rightward and vertical rays directed upward (no two being colinear)?

Exercise 2. Let n be a positive integer. A *Baxter permutation* is a permutation of $\{1, \ldots, n\}$ which avoids the pattern 2-41-3 and 3-14-2. That is, π is Baxter if there are no $i, j, k \in [n]$ with $i < j < j + 1 < k$ so that $\pi_j < \pi_k < \pi_i < \pi_{j+1}$ nor $\pi_{j+1} < \pi_i < \pi_k < \pi_j$.

Given a rectangulation of a square with n rectangles, consider its elimination sequence towards the top-left corner and label the rectangles in the order they disappear. See Figure below. Now consider the elimination sequence towards the bottom-left corner and list the rectangles using their labels from the top-left ordering.

Prove that the set of permutations of $[n]$ you can obtain over all rectangulations of order n is exactly the set of all Baxter permutations of $[n]$.



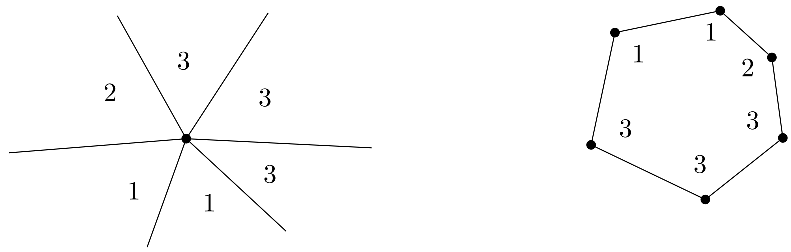
Exercise 3. Prove that a poset of height 2 with a planar cover graph has dimension at most 4.

Exercise 4. Prove that a poset with a planar comparability graph has dimension at most 4. *Hint*: First reduce to the case when the poset is of height at most 3 and every element at the middle level is above exactly two minimas and below exactly two maximas. To do that you do dirty tricks and might want to use the split operation applied to some selected elements. Then, try to interpret the poset as a subposet of the vertex-edge-face poset of a planar multigraph.

Exercise 5. A *planar map* M is a simple planar graph G together with a fixed planar embedding of G in the plane. A *suspension* M^σ of M is obtained by selecting three different vertices a_1, a_2, a_3 in clockwise order from the outer face of M and adding a half-edge that reaches into the outer face to each of these special vertices.

Let M^σ be the suspension of a 3-connected planar map. A *Schnyder angle labelling* with respect to a_1, a_2, a_3 is a labeling of the angles of M^σ with the labels 1, 2, 3 (alternatively: red, green, blue) satisfying three rules:

(i) The two angles at the half-edge of the special vertex a_i have labels $i+1$ and $i-1$ in clockwise order.
(ii) The labels of the angles at each vertex form, in clockwise order, nonempty intervals of 1's, 2's and 3's.
(iii) The labels of the angles at each face form, in clockwise order, a nonempty interval of 1's, a nonempty interval of 2's and a nonempty interval of 3's. At the outer face the same is true in counterclockwise order.



Show that for every 3-connected planar map M with a suspension M^σ, the Schnyder woods of M^σ are in bijection with the Schnyder angle labellings of M^σ.

Exercise 6. Consider a quadrangulation G suspended on two opposite vertices a_1, a_2 on the outer face. Let $\mathcal{S}$ be a separating decomposition of G. For each vertex x in G, define two regions $R_1(x)$ and $R_2(x)$ as an analogue to regions of a Schnyder wood. Let $f_i(x)$ be the number of (bounded) faces of G in $R_i(x)$, for $i \in \{1, 2\}$. Show that $f_i(x)$ corresponds to the position of x on the equatorial line of $\mathcal{S}$.