Homework 7

Due to November 26, 23:59.

Warm-up. Just think about them. Do not submit.

Exercise. Let $<_0, <_1, \ldots, <_t$ be a set of linear orders of $[n]$ such that for $x, y, z \in [n]$ there is an i such that $y, z <_i x$. Define $S_{x,y} = \{i \ : \ 1 \leqslant i \leqslant t \text{ with } x <_i y\}$ and $A_x = \textsc{Max}\{S_{x,y} \ : \ x <_0 y\}$. Show that $A_x \neq A_y$ for all $x \neq y$.

Exercise. Show that $\dim(s', t'\,; n') \leqslant \dim(s, t\,; n)$ for all $s \leqslant s' < t' \leqslant t < n' \leqslant n$.

Exercise. Let P be an interval order and I be its interval representation (with no interval degenerated to a point). Show that for every linear extension L there is a marking function f of I such that for all x, y in P we have $m(x) < m(y)$ iff $x < y$ in L.

Please submit solutions of the following. To be graded.

Exercise 1. Let $s + t \leqslant n$ and $\lfloor t/s \rfloor + s - 1 \leqslant k$. Show that $\dim(1, k\,; n) > t$.

The *fractional dimension* of P is the minimum $t \in \mathbb{R}$ such that there is a multi-realizer $\mathcal{R}$ of P such that $t = \max\big(\frac{|\mathcal{R}|}{|\mathcal{R}(a<b)|} : (a, b) \in \mathrm{inc}(P)\ \big)$. Here $\mathcal{R}(a < b) = \{L \in \mathcal{R} : a <_L b\}$, a multi-realizer is a multi-set realizer, and 'minimum t' can be replaced by 'infimum of all t' if you worry. Remark: It may be convenient to equip $\mathcal{R}$ with the uniform distribution, in this setting $1/t = \min\big(\mathrm{Prob}(a < b) : (a, b) \in \mathrm{inc}(P)\ \big)$.

Exercise 2. Determine the fractional dimension of $\mathcal{B}_n(1, t)$, i.e., of levels 1 and t in the Boolean lattice $\mathcal{B}_n$.

Exercise 3. Determine good upper bound for the fractional dimension of interval orders.

Exercise 4. Let $I(k, s)$ be the set of all interval orders with $k+s-1$ elements, a connected comparability graph, and exactly k maximal antichains, each of them of size s.
Let $a(k, s) = |I(k, s)|$. Determine the values of $a(16, 6)$ and $a(19, 7)$ and $a(30, 2)$.

Exercise 5. [`0.5 pt`] Let $N(m)$ be the minimal n such that the width of $\mathcal{B}_n$ is at least m. Let P be an interval order of width w. Show that

$$\dim(P) \leqslant N(w).$$

Exercise 6. [`0.5 pt`] Let P be an interval order which has no induced $\mathbf{t + 1}$. Show that

$$\dim(P) \leqslant N(t - 1) + 1.$$

Exercise 7. Let P be an interval order of height h. Show that

$$\dim(P) \in O(\log\log(h)).$$