Homework 8

Due to December 3, 23:59.

Warm-up. Just think about them. Do not submit.

Exercise. Prove that overlap graphs are χ-bounded. *Hint*: Layer the vertices with respect to their distance to the root. Recursively. A lemma given within Lecture 15 might be helpful. *Comment:* This kind of proof gives an exponential upper bound on χ in terms of ω. For many years this was the state of art and only recently there was a major progress: arXiv:1905.11578.

Exercise. Assume that the class of intersection graphs of grounded segments is χ-bounded. (This is true but perhaps we will not go over the whole proof.) Building on that show that the class of intersection graphs of unit-length segments in the plane is χ-bounded.

A thrackle is an embedding of a graph in the plane, such that each edge is a Jordan arc and every pair of edges meet exactly once. Edges may either meet at a common endpoint, or, if they have no endpoints in common, at a point in their interiors (in that case the crossing must be transverse).

Exercise. Find thrackle embeddings of:

(i) odd cycles;
(ii) even cycles of length at least 6 (4-cycle has no thrackle embedding);
(iii) trees.

Exercise. For a given family $\mathcal{F}$ of axis-aligned rectangles in the plane, consider a poset $P(\mathcal{F})$ whose elements are these rectangles and $x < y$ in $P(\mathcal{F})$ iff the rectangle of x crosses the rectangle of y and the rectangle of x has larger x-projection than the rectangle of y. What is the maximum of $\dim(P(\mathcal{F}))$?

Please submit solutions of the following. To be graded.

Exercise 1. Consider disjointness graphs of axis-aligned rectangles in the plane. Prove that $\chi = \mathcal{O}(\omega \log \omega)$ for graphs in this class.

Exercise 2. Show that given a family of n convex compact sets in the plane, one can always find $n^{1/5}$ of them which are either pairwise intersecting or pairwise disjoint.

Exercise 3. Show that a convex geometric graph (vertices are the corners of a convex polygon) with $n \geqslant 2k+1$ vertices which has no $k+1$ pairwise crossing edges can have at most $k(2n-2k-1)$ edges. (This bound is actually tight).

Exercise 4. First, show that if R is a circular sequence with entries from a set of $n > 1$ symbols such that no two adjacent entries are identical and R contains no circular subsequence of type $abab$, then the length of R is at most $2n - 2$. (Such a sequence is called a circular Davenport-Schinzel sequence of order 2. Davenport-Schinzel sequences are a strong combinatorial tool to bound complexities in geometric settings). Now, two edges in a geometric graph are said to be parallel, if they are two opposite edges of a convex quadrilateral. Show that a geometric graph without parallel edges can have at most $2n - 2$ edges.

Let c be an integer and $c \geqslant 1$. Fix c disjoint horizontal lines. Within Lecture 16, a c-*interval* is defined to be a union of c intervals one from each of the fixed lines. We were considering intersection graphs of these objects. So an independent set is just a set of pairwise disjoint c-intervals. We have seen that for every $t \geqslant 1$ and $c \geqslant 1$, there exists an integer $f(c,t)$ such that for every family $\mathcal{F}$ of c-intervals with no independent set of size t, we can find $f(c,t)$ points that pierce each c-interval.

A c-**interval* is a union of c intervals on a *single* line.

Exercise 5. Prove that for every fixed $c \geqslant 1$, the class of intersection graphs of c-*intervals is χ-bounded. Try to get a good bound in terms of ω.

Exercise 6. Show that for every $t \geqslant 1$ and $c \geqslant 1$, there exists an integer $g(c,t)$ such that every family $\mathcal{F}$ of c-*intervals with no independent set of size t can be pierced by at most $g(c,t)$ points.