Homework 9

Due to December 10, 23:59.

Warm-up. Just think about them. Do not submit.

Exercise. Suppose that we are given a 3-colorable graph G but we do not know a witnessing coloring. Show how to color G with $\mathcal{O}(\sqrt{n})$ in polynomial time.

Exercise. A *circle order* is a containment order of discs in the plane. Show that standard examples are circle orders.

Please submit solutions of the following. To be graded.

Exercise 1. Construct a family of arbitrarily large triangle-free graphs with the chromatic number in $\Omega(n^{1/3})$ where n is the number of vertices. Follow the description.

Consider a finite projective plane of order q. So with q^2+q+1 points and q^2+q+1 lines. (You can browse the web for additional information on this object.) Fix an arbitrary ordering $\prec$ on points of the plane and an arbitrary ordering $\prec$ on lines of the plane Consider a graph G_q with vertices being all the point-line incidences. Two vertices (p_1, ℓ_1) and (p_2, ℓ_2) are adjacent in G_q if $p_1 \prec p_2$, $\ell_1 \prec \ell_2$ and (p_1, ℓ_2) is a point-line incidence (so is a vertex of G_q). Prove that

(i) G_q is triangle-free;
(ii) if I is an independent set in G_q then reading I as the set of edges in the bipartite graph of point-line incidences, we have that I is acyclic.

Conclude the bound using a standard inequality $\chi \geqslant \frac{n}{\alpha}$ where α is the size of the largest independent set.

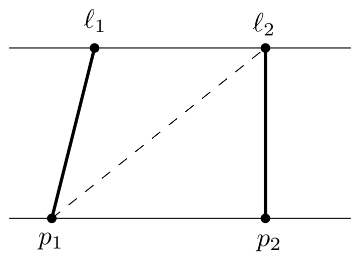

Figure 1: Two adjacent point-line incidences in G_q. The order of points and lines is depicted with two horizontal lines.

Exercise 2. Prove that d-dimensional posets are containment orders of simple polygons with d-corners. Can you make the polygons convex?

Exercise 3. Let P be a poset and let $\mathcal{R}$ be its realizer. When we fix an ordering on linear extensions in $\mathcal{R}$ we say that $\mathcal{R} = (L_1, \ldots, L_k)$ is an *ordered realizer*. We say that an (incomparable) pair (x, y) of elements in P *flip at coordinate* i ($1 \leqslant i < k$) if ($x < y$ in L_i and $y < x$ in L_{i+1}) or ($y < x$ in L_i and $x < y$ in L_{i+1}). The *crossing number* of an ordered realizer is the maximum over all pairs of elements of P of the number of times the pair flips. The *crossing number* of a poset is the minimum crossing number of its realizer. Clearly, d-dimensional orders have crossing number of at most $d - 1$. Show that this bound is tight.

Exercise 4. Show that the class of disjointness graphs of families of grounded x-monotone curves is χ-bounded. *Hint or just comment*: $\chi \leqslant \binom{\omega+1}{2}$.

Exercise 5. Show that for every integer $k \geqslant 3$, every 3-dimensional order is a containment order of regular k-gons. (It is a deep result that there is a 3-dimensional order that is not a circle order.)

Exercise 6. Read and digest [arXiv] a short argument that shift graphs are disjointnesss graphs of 1-intersecting curves in the plane.

Exercise 7. Suppose that a graph G is a disjointness graph of curves spanned in a cylinder Show that G is a weakly transitive orientation. (Consult Lecture 18.)