Homework 10

Due to December 17, 23:59.

Warm-up. Just think about them. Do not submit.

Exercise. What are the facets of the chain polytope $\mathcal{C}(P)$?

Exercise. For a graph $G = (V, E)$ we define

$$\mathcal{C}(G) = \{a \in [0,1]^V \mid \sum_{v \in C} a_v \leqslant 1 \quad \text{for every clique } C \text{ of } G\}.$$

If P is an order with comparability graph G, then $\mathcal{C}(P) = \mathcal{C}(G)$.

Show that in general $\mathcal{C}(G)$ may have corners which are not characteristic vectors of stable sets.

Please submit solutions of the following. To be graded.

Exercise 1. Consider the independence number of a cover graph of randomly and uniformly selected n-element point set in the unit square. Let X be the random variable counting number of edges in this graph.

(i) Show that $\mathrm{E}(X) = \Theta(n \log n)$.
(ii) Show that $\mathrm{Var}(X) = \mathcal{O}(n \log^2 n)$.
(iii) Apply Chebyshev's inequality and show that for every sufficiently large n there is a point set P (actually with high probability) generating a cover graph with $\Theta(n \log n)$ edges. Conclude that the cover graph of P has independence number $\Omega\left(\frac{n}{\log n}\right)$,

Exercise 2. For a poset P, let $m(P)$ be the maximum integer m such that there are two disjoint subsets A and B of elements of P with $|A| = |B| = m$ and $\forall_{a \in A,\ b \in B}\ a < b$ in P or $\forall_{a \in A,\ b \in B}\ a \parallel b$ in P. Show that for every poset P with n elements and dimension d, we have

$$m(P) \geqslant \left\lfloor \frac{n}{2d} \right\rfloor.$$

Exercise 3. Show that the maximum number of edges of a $K_{t,t}$-free incomparability graph of a 2-dimensional poset with n elements is at most

$$2(t-1)n - \binom{2t-1}{2},$$

for every $t \geqslant 2$ and $n \geqslant 2t - 1$.
Hint: Show that such graphs are $(2t-2)$-degenarate.

Exercise 4. Prove that there is a (finite) point set P in the plane such that no matter how we two-color points in P there is always an axis-aligned rectangle containing exactly 2020 points from P all of which of the same color.

Exercise 5. Identify the faces of the order polytope $\mathcal{O}(P)$, i.e., describe a class of combinatorial objects which are in bijection to the faces. *Hint*: It may be helpful to think of $\hat{P}$, i.e., of P enriched with additional global $\mathbf{0}$ and $\mathbf{1}$ elements.

Exercise 6.

(i) What is the maximum of $e(P)$ over all posets with n elements and width at most w?

(ii) What is the minimum of $e(P)$ over all posets with n elements and height at most h?