Homework 11

Due to January 7, 23:59.

Please submit solutions of the following. To be graded.

Exercise 1. Let P be a ranked poset. That is, there is a function $r : P \to \mathbb{N}$ such that $r(x) = 0$ for all minimal elements x in P and $r(y) = r(x) + 1$ whenever y covers x in P. Let h be the height of P and $N_i = |\{x \in P \mid r(x) = i\}|$ for all $i \in \{0, \ldots, h-1\}$. We say that P has the *LYM property*, if for every antichain A in P we have

$$\sum_{i=0}^{h-1} \frac{|\{a \in A \mid r(a) = i\}|}{N_i} \leqslant 1.$$

Show that if P has the LYM property, then

$$\prod N_i! \leqslant e(P) \leqslant \prod N_i^{N_i}.$$

Exercise 2. Let P_1, P_2, P_3 be orders on the same set X of elements, and let G_1, G_2, G_3 be their comparability graphs. We aim at showing that if $E(G_1) \cap E(G_2) \subseteq E(G_3)$, then $e(P_1) \cdot e(P_2) \geqslant e(P_3)$.

(i) Let $a_x = e(P_1 - x)/e(P_1)$ if $x \in \mathrm{Min}(P_3)$ and $a_x = 0$ otherwise. Show that $a \in \mathcal{C}(P_2)$.
(ii) Let $b_x = e(P_2 - x)/e(P_2)$ for all x and show that $b \in \mathcal{A}(P_2)$.
(iii) Use $a^\top b \leqslant 1$, induction, and the inequality $e(P_3) \leqslant \sum_{x \in \mathrm{Min}(P_3)} e(P_3 - x)$ to finally show $e(P_3) \leqslant e(P_1) \cdot e(P_2)$.
(iv) Use the previous to show that for a 2-dimensional P with conjugate $\bar{P}$ we have

$$e(P) \cdot e(\bar{P}) \geqslant n!.$$

Exercise 3. In this little project we aim at a probabilistic proof of the hook-formula for trees. Let T be a poset with n elements such that its diagram is a rooted tree and the element corresponding to the root is below all other elements (so the set of leaves coincides with $\mathrm{Max}(T)$). Consider the following algorithm:

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random-max(T)
    x ← uniform random vertex from T
    while x ∉ Max(T)
        x ← uniform random vertex from U(x)
    return x
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(i) Let $y \in \mathrm{Max}(T)$, show that

$$\mathsf{Prob}(y = \mathsf{random\text{-}max}(T)) = \frac{1}{n} \prod_{z<y} \frac{h_z}{h_z - 1}.$$

(ii) Let $F(T) = \frac{n!}{\prod h_x}$. Use the previous item to show that $F(T) = \sum_{y \in \mathrm{Max}(T)} F(T - y)$ and conclude that $F(T) = e(T)$.

Exercise 4. Let $\lambda = \{\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_m\}$ be a partition of a positive integer n. The *Ferrers diagram* of λ is an array of cells doubly indexed by pairs (i, j) with $1 \leqslant i \leqslant m$, $1 \leqslant j \leqslant \lambda_i$. We might like to look at the diagram as a poset with $(i, j) \leqslant (i', j')$ iff $i \leqslant i'$ and $j \leqslant j'$. A *Young tableau of shape* λ is an arrangement of integers $1, \ldots, n$ in the cells of the Ferrers diagram of λ such that all rows and all columns form increasing subsequences. Thus, it is nothing else but a linear extension of the corresponding poset. The total number of Young tableaux of shape λ will be denoted by f_λ.

For each cell (i, j) define the *hook* H_{ij} to be the collection of cells (a, b) such that $a = i$ and $b \geqslant j$ or $a \geqslant i$ and $b = j$. Define $h_{ij} = |H_{ij}|$.

This little project is to prove that for every λ partition of n we have

$$f_\lambda = \frac{n!}{\prod h_{ij}},$$

where the product is over all cells in the Ferrers diagram of λ. For convienience, let $F(\lambda)$ be the formula on the right-hand side above.

For example, if $\lambda = \{3, 2\}$ and $n = 5$, the hook lengths of each cell in the Ferrers diagram of λ are as shown:

$$\begin{matrix} 4 & 3 & 1 \\ 2 & 1 & \end{matrix}$$

According to the theorem above, the number of Young tableaux of shape λ is equal $\frac{5!}{4 \cdot 3 \cdot 1 \cdot 2 \cdot 1} = 5$, which can be easily verified.

Consider the following algorithm:

random-corner(λ)
  $(i, j) \leftarrow$ uniform random cell of the Ferrers diagram of λ
  while (i, j) is not a corner cell
    $(i, j) \leftarrow$ uniform random cell from $H_{ij} \setminus \{(i, j)\}$
  return (i, j)

(i) Let (α, β) be a corner of the Ferrers diagram of $\lambda = (\lambda_1, \ldots, \lambda_m)$. Let $\lambda_\alpha^- = (\lambda_1, \ldots, \lambda_{\alpha-1}, \lambda_\alpha - 1, \lambda_{\alpha+1}, \ldots, \lambda_m)$. Note that λ_α^- is a partition of n as well (as (α, β) is a corner cell). Show that

$$\mathsf{Prob}((\alpha, \beta) = \mathsf{random\text{-}corner}(\lambda)) = \frac{F(\lambda_\alpha^-)}{F(\lambda)}.$$

(ii) Use the previous item to conclude the hook formula.

Exercise 5. Consider the following method of shuffling n cards. At each step, a card is chosen independently and uniformly at random and put on the top of the deck. We can think of the shuffling process as a Markov chain, where the state is the current order of the cards. Clearly, the chain is finite, irreducible, and aperiodic, so it has a unique stationary distribution that is uniform over all the states.

How long do we have to shuffle to get close to the uniform distribution? A chain is *rapidly mixing* if $\tau(\varepsilon)$ is polynomial in $\log(1/\varepsilon)$ and the size of the problem (the number of cards).

(i) Consider the following coupling (X_t, Y_t) of the described chain. Choose a position j uniformly at random from 1 to n and then obtain X_{t+1} from X_t by moving the jth card to the top. Denote the value of this card by C. To obtain Y_{t+1} from Y_t move the card with value C to the top.
Argue that the coupling is valid and using it to show that the discussed chain is rapidly mixing.

(ii) Forget about the coupling and prove that at the moment that each card in the chain was picked and put at the top at least once the distribution of the current state is uniform over all permutations.

Exercise 6. Consider the following variation on shuffling of a deck of n cards. At each step, a specific card is chosen uniformly at random from the deck, and a position $i \in \{1, \ldots, n\}$ is chosen uniformly at random; then the card at position i exchanges positions with the specific card chosen.

Devise a coupling of this chain such that the expected number of steps until the two copies coincide is $\mathcal{O}(n^2)$.

Exercise 7. Construct a family of posets $\{P_n\}$ with $|P_n| = n$ such that the Bubley-Dyer chain has a mixing time $\Omega(n^3)$.