Homework 12

Due to January 21, 23:59.

Please submit solutions of the following. To be graded.

Exercise 1.

(i) Show that for every $h \geqslant 1$ there is a poset P of height at most h with a planar diagram such that
$$\dim(P) \geqslant (4/3)h - 2.$$

(ii) Show that for every $h \geqslant 1$ there is a poset P of height at most h with a planar cover graph such that
$$\dim(P) \geqslant 2h - 2.$$

Hint: See Figures 1 and 2.

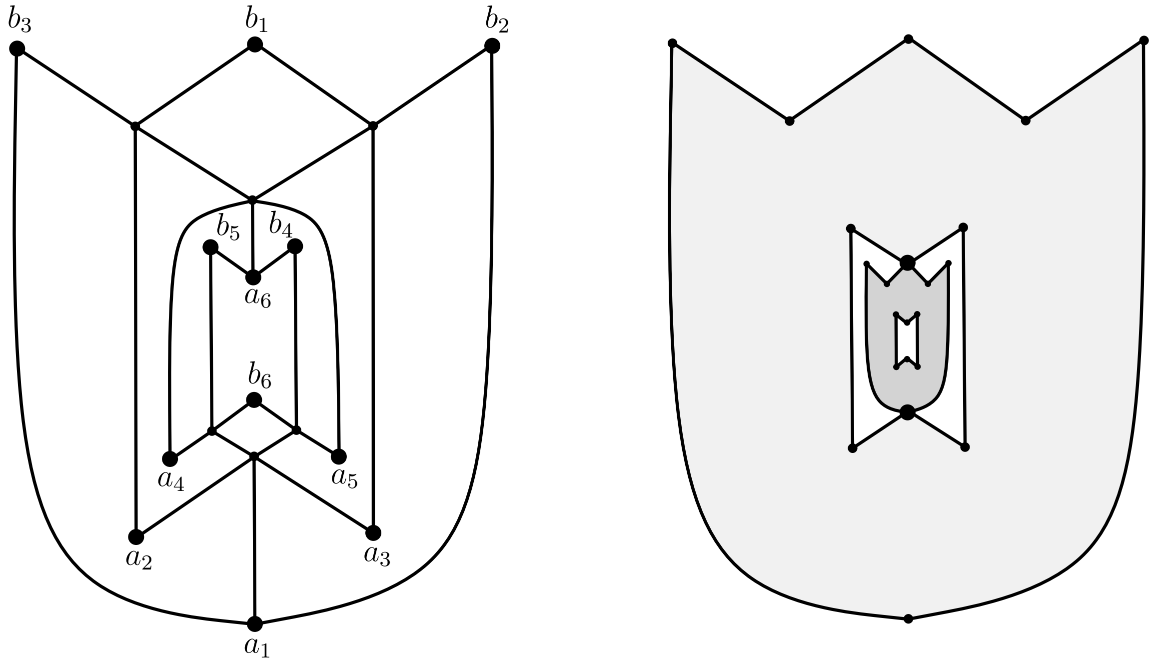

Figure 1: Iterative construction of posets with planar diagrams and arbitrarily large dimension.

Exercise 2.

(i) Let T be a forest. What is the maximum possible value of $\mathrm{wcol}_r(T)$?
(ii) Let T be a complete binary tree. What is the maximum possible value of $\mathrm{wcol}_r(T)$?

Give both: upper bounds and lower bounds.

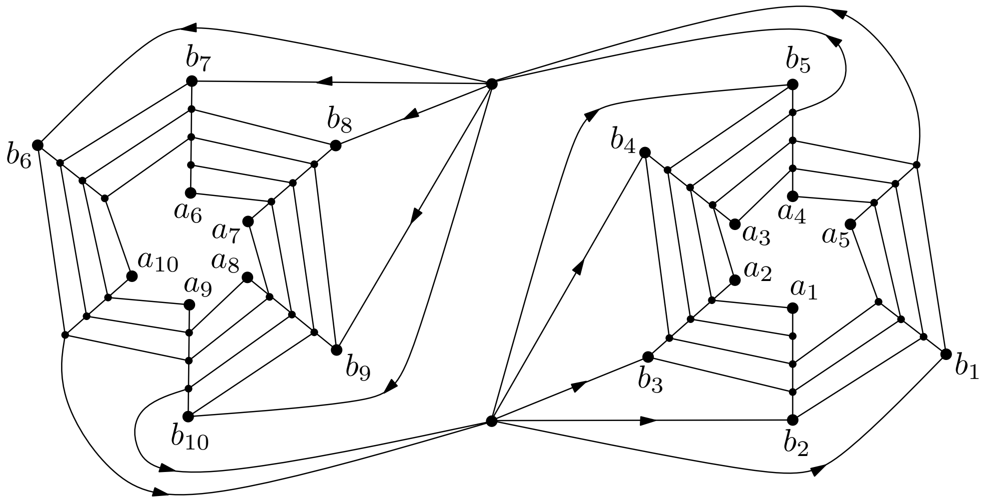

Figure 2: Construction of posets with planar cover graphs and large dimension.

Let G be a graph and π be a linear order on $V(G)$. For $v \in V(G)$ and $r \geqslant 0$ we say that $u \in V(G)$ ist *r-strongly reachable* from v w.r.t. π in G if there is a path Q from v to u in G such that u is the minimum vertex of Q in π and for all internal vertices w of Q we have $w > v$ in π. We define

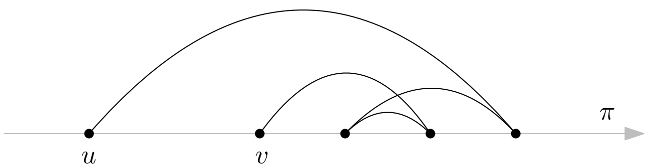

Figure 3: u is 4-strongly reachable from v w.r.t. π.

$$\mathrm{SReach}_r^\pi = \{u \in V(G) \mid u \text{ is } r\text{-strongly reachable from } v \text{ in } \pi\},$$
$$\mathrm{scol}_r(G) = \min_\pi \max_{v \in V(G)} |\,\mathrm{SReach}_r^\pi(v)|.$$

Exercise 3.

(i) Show that for every graph G and every integer $r \geqslant 0$ we have

$$\mathrm{scol}_r(G) \leqslant \mathrm{wcol}_r(G) \leqslant (\mathrm{scol}_r(G))^r.$$

(ii) The *grid* $\boxplus_n$ of order n is the graph with the vertex set $\{(i,j) \mid i,j \in \{1,\ldots,n\}\}$ whose two distinct vertices (i,j), (i',j') are adjacent iff $|i'-i| + |j'-j| = 1$. What are the possible asymptotics for functions f and g such that

$$f(r) \leqslant \mathrm{scol}_r(\boxplus_n) \leqslant \mathrm{wcol}_r(\boxplus_n) \leqslant g(r),$$

where n goes over all the natural numbers?

Exercise 4. Let G be a graph and p be an integer with $p \geqslant 1$. The *exact p-distance* graph $G^{[\#p]}$ is the graph with the same vertex set as G and two vertices are adjacent in $G^{[\#p]}$ if they are of distnace exactly p in G. Show that whenever p is odd we have

$$\chi(G^{[\#p]}) \leqslant \mathrm{wcol}_{2p-1}(G).$$

Hint: Consider an auxilliary greedy coloring along an ordering witnessing $\mathrm{wcol}_{2p-1}(G)$. To devise the final coloring consider balls $N^{\lfloor p/2 \rfloor}(v)$ in G, i.e. all vertices of distance at most $\lfloor p/2 \rfloor$ from v in G.

Exercise 5. Let G be an interval graph with $\omega(G) \leqslant k$. Prove that for every $r \geqslant 0$

$$\mathrm{wcol}_r(G) \leqslant \binom{r+k-1}{r}$$

and that this bound is tight (for all $r \geqslant 0$ and $k \geqslant 1$).

Exercise 6. Let $d \in \mathbb{N}$, let G be a graph, and let σ be a vertex ordering of G. Consider the following algorithm. Every vertex $u \in V(G)$ picks $v(u)$ to be the smallest vertex of $\mathrm{WReach}_d[G, \sigma, u]$ in σ. Then, define D as the set of those vertices that have been picked by any vertex:

$$D := \{v(u) \mid u \in V(G)\}.$$

Prove that D is a distance-d dominating set of G that satisfies $|D| \leqslant \mathrm{wcol}_{2d}(G, \sigma) \cdot \mathrm{dom}_d(G)$.

Comment: A *distance-d dominating set* in G is a subset of vertices of G such that every vertex v in G is distance at most d to some vertex of the subset. $\mathrm{dom}_d(G)$ is the minimum size of a distance-d dominating set in G.