Examples, chains, and antichains

Exercise 1.

(i) How many edges has the diagram of the Boolean lattice $\mathcal{B}_n$?
(ii) How many edges has the comparability graph of $\mathcal{B}_n$?
(iii) How many 3-chains does the Boolean lattice $\mathcal{B}_n$ have?
(iv) How many 2-antichains does the Boolean lattice $\mathcal{B}_n$ have?

Exercise 2. Given a collection of orders $(X, \leqslant_i)$ on a common ground-set X. Show that $(X, \bigcap_i \leqslant_i)$ is again a partial order.

Exercise 3.

(i) Let G be a comparability graph, show that G contains no odd cycle of length $\geqslant 5$ as induced subgraph.
(ii) Let G be a incomparability graph, show that G contains no odd cycle of length $\geqslant 5$ as induced subgraph.

Exercise 4. Let G be a bipartite graph, how many transitive orientations does G have?

Exercise 5. Let G be a graph with a girth $\mathrm{g}(G)$. Show that if $\mathrm{g}(G) > \chi(G)$, then G is a cover graph.

Exercise 6. Let P be an order. Characterize the pairs (x, y) with the property that

∗ $P + (x < y)$ is an order,
∗ $P - (x < y)$ is an order.

Exercise 7. For each m construct a partial order P_m with $\binom{m+1}{2}$ elements such that if $B_1, \ldots, B_k$ is a cover of P_m with the property that each B_i is a chain or an antichain, then $k \geqslant m$. (Later we will see that Greene-Kleitman Theory implies that every order with less than $\binom{m+1}{2}$ elements has such a cover with $k < m$.)

A poset P is a *weak order* if the elements of P can be partitioned into antichains $A_1, \ldots, A_n$ $(n \geqslant 1)$ in such a way that $a_i < a_j$ in P for all $1 \leqslant i < j \leqslant n$, $a_i \in A_i$, $a_j \in A_j$.

Exercise 8. Show that the following two conditions are equivalent

(i) P is a weak order,
(ii) P is $(2+1)$-free.

Exercise 9. Let I be a family of n intervals on the real line. Show that either I contains $\lceil\sqrt{n}\rceil$ pairwise disjoint intervals or I contains $\lceil\sqrt{n}\rceil$ intervals that all share a common point.

Exercise 10. Find an infinite partial order that has no infinite antichain but is not a union of finitely many chains.