Linear extensions, dimension, and Boolean lattices

Exercise 1. Prove that the following conditions are equivalent:

(i) G is a comparability graph of a poset of dimension at most 2;
(ii) G is a containment graph of intervals on a line;
(iii) G is a permutation graph;
(iv) G and its complement are both comparability graphs.

Exercise 2. Let $\dim^*(P)$ be the least integer d such that the elements of G can be embedded into $\mathbb{R}^n$ in such a way that for every x, y in P we have $x \leqslant y$ in P if and only if the point of x is less or equal the point of y in the product order on $\mathbb{R}^n$. Prove that $\dim(P) = \dim^*(P)$.

Exercise 3. Let $P = (X, \leqslant)$ be a poset. For a linear extension L of P, let $s(L)$ be a string over X with symbols aligned as elements in L. Prove that the set

$$\{s \mid s \text{ is a prefix of } s(L) \text{ for some linear extension } L \text{ of } P\}$$

is an antimatroid over X.

Exercise 4. Let P be a poset and x be an element of P. Let L be a linear extension of $P - x$. Show that one can always extend L to L^+ introducing x so that L^+ is a linear extension of P.

Exercise 5. Let P and Q be the posets and let $\dim(P) = d$. Show that

(i) $\dim(P \smallsetminus \{x\}) \in \{d-1, d\}$ for every $x \in P$,
(ii) $\dim(P \smallsetminus \{x, y\}) \in \{d-1, d\}$ for every $x \in \min(P)$, $y \in \max(P)$, $x||y$,
(iii) $\dim(P + Q) \leqslant \max(\dim(P), \dim(Q), 2)$,
(iv) $\dim(P \times Q) \leqslant \dim(P) + \dim(Q)$.

Exercise 6. Let P be a poset and C be a chain in P. Prove that

$$\dim(P) \leqslant \dim(P - C) + 2.$$

Exercise 7. Let M be a subset of maximal elements of a poset P. Let $\text{width}(P \smallsetminus M) \leqslant w$. Show that

$$\dim(P) \leqslant w + 1.$$

Exercise 8. A poset is 3-*irreducible* if it has dimension 3 and after removing any element the dimension drops to 2. There is a complete list of 3-irreducible posets (it includes some infinite families). Below we present some posets from the list. Prove that the dimension of posets below is at least 3.

(i) The *crown* C_n of order n is a poset on $2n$ elements $x_1, \ldots, x_n, y_1, \ldots, y_n$ with $x_i < y_i$ and $x_i < y_{i+1}$ for $i \in \{1, \ldots, n\}$ (cyclically) and no other strict comparabilities. See Figure 1.

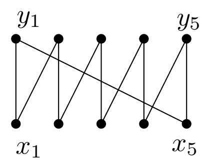

Figure 1: The crown of order 5.

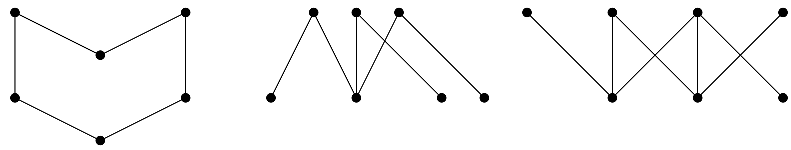

Figure 2: Some sporadic examples.

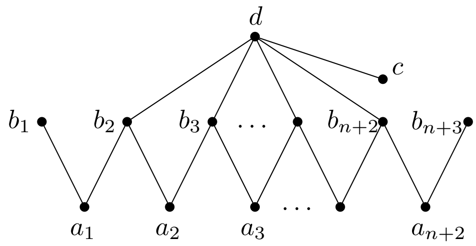

Figure 3: The family of posets $\{Q_n\}_{n \geqslant 0}$.

(ii) See Figure 2 for some sporadic examples: the chevron, the spider, and one more.
(iii) See Figure 3.

Exercise 9. Consider a symmetric chain decomposition $\mathcal{C}$ on the Boolean lattice $\mathcal{B}_n$. How many chains of size k are in $\mathcal{C}$ for $1 \leqslant k \leqslant n + 1$?

Exercise 10. Show that, if $A_1, \dots A_m$ are distinct k-subsets of an n-set and $k \leqslant s \leqslant n-k$, then there exist distinct s-subsets $B_1, \dots, B_m$ such that $A_i \cap B_i = \emptyset$ for each $i = 1, ..., m$.