Boolean lattices and intersecting families

Exercise 1. What is the largest size of an antichain in $\mathcal{B}_n$ with at least one set of size at most 2, at least one set of size at least $n-2$, and no sets of size i, for all $i \in \{3, \dots, n-3\}$.

Exercise 2. Prove that the families $\binom{[n]}{\lfloor n/2 \rfloor}$ and $\binom{[n]}{\lceil n/2 \rceil}$ are the only antichains in $\mathcal{B}_n$ of size $\binom{n}{\lceil n/2 \rceil}$.

Exercise 3 (Littlewood Offord Problem)**.** Let $a_1, \dots, a_n$ be a sequence of reals such that $|a_i| \geqslant 1$ for all $i \in [n]$. Let

$$P(a_1, \dots, a_n) = \{(\varepsilon_1, \dots, \varepsilon_n) \in \{-1, 1\}^n \mid -1 < \sum_{i=1}^{n} \varepsilon_i \cdot a_i < 1\}.$$

Show that $|P(a_1, \dots, a_n)| \leqslant \binom{n}{\lfloor n/2 \rfloor}$.

Exercise 4. Let $\mathcal{F}$ be a family of subsets of $[n]$ with no $A_1, \dots, A_s \in \mathcal{F}$ such that $A_1 \subsetneq A_2 \subsetneq \cdots \subsetneq A_s$. Prove that the size of $\mathcal{F}$ is at most the sum of s largest binomials of the form $\binom{n}{i}$ for any integer i.

Exercise 5. Let $1 \leqslant s < r < n$ and let $\mathcal{F}$ be a family of r-subsets of $[n]$ such that for every $A \neq B \in \mathcal{F}$ we have $|A \cap B| \leqslant s$. Show that

$$|\mathcal{F}| \leqslant \frac{\binom{n}{s+1}}{\binom{r}{s+1}}.$$

Exercise 6. Fix $1 \leqslant k \leqslant n \leqslant 2k$ and show that if $\mathcal{F}$ is a family of subsets of $[n]$ each of size at least k, and not containing two sets whose union is $[n]$ then $|\mathcal{F}| \leqslant \binom{n-1}{k}$.

Exercise 7. Let S_i be the shifting operation from Lecture 5. Suppose that $\mathcal{F}$ is intersecting, show that this implies that $S_i(\mathcal{F})$ is intersecting.

Exercise 8. Let σ be a cyclic permutation of $[n]$ and let S be a family of k-arcs of σ with $\triangle_\sigma(S)$ we denote the σ-shadow of S, i.e., the set of all $(k-1)$-arcs contained in an arc of S. Show that unless $|S| = n$ we have $| \triangle_\sigma (S)| > |S|$.

Exercise 9. let $\mathcal{A}$ be a family of k-subsets of $[n]$ such that for every h tuple $(A_1, A_2, \dots, A_h)$ of sets from $\mathcal{A}$ we have $A_1 \cap A_2 \cap \dots \cap A_h \neq \emptyset$. Show that if $k \cdot h \leqslant (h-1)n$, then $|\mathcal{A}| \leqslant \binom{n-1}{k-1}$.

Exercise 10. Let $F_k(m)$ be the collection of the first m sets in the colex order on k-sets. Show that for $m \geqslant 1$ we have $| \triangle F_k(m+1)| \leqslant | \triangle F_k(m)| + (k-1)$.

Exercise 11. Show $\triangle F_k(m) = F_{k-1}(m')$ for some m'.