Some of My Favorite Combinatorial Problems for Posets

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(1) First Fit Coloring of Interval Graphs

Problem (Woodall, '76) Let FF(k) be the maximum number of colors First Fit can be forced to use on an interval graph with maximum clique size k.

Known bounds:

$$(5-o(1))\,k \le FF(k) \le 8k - 4$$

The lower bound is due to Kierstead, D. Smith and WTT. The upper bound is due (essentially) to Pemmaraju, Raman and Varadarajan, with minor improvements by (1) Brightwell, Kierstead and WTT, and (2) Narayansamy and Babu; and (3) Howard.

(2) k-Crossing Vectors

Definition For positive integers k, w, two vectors A and B in Z^w are said to be **k-crossing** if there are coordinates i and j with $A(i) \geq k + B(i)$ and $B(j) \geq k + A(j)$.

Problem Find the maximum size $f(k, w)$ of an antichain in Z^w which contains no k-crossing pair.

It is known that:

$$k^{w-1} \leq f(k, w) \leq k^w$$

Note Lasoń, Micek, Streib, WTT and Walczak proved that the lower bound is tight when $w \leq 3$ and conjecture that $f(k, w) = k^{w-1}$.

(3) Monotone Hamiltonian Paths

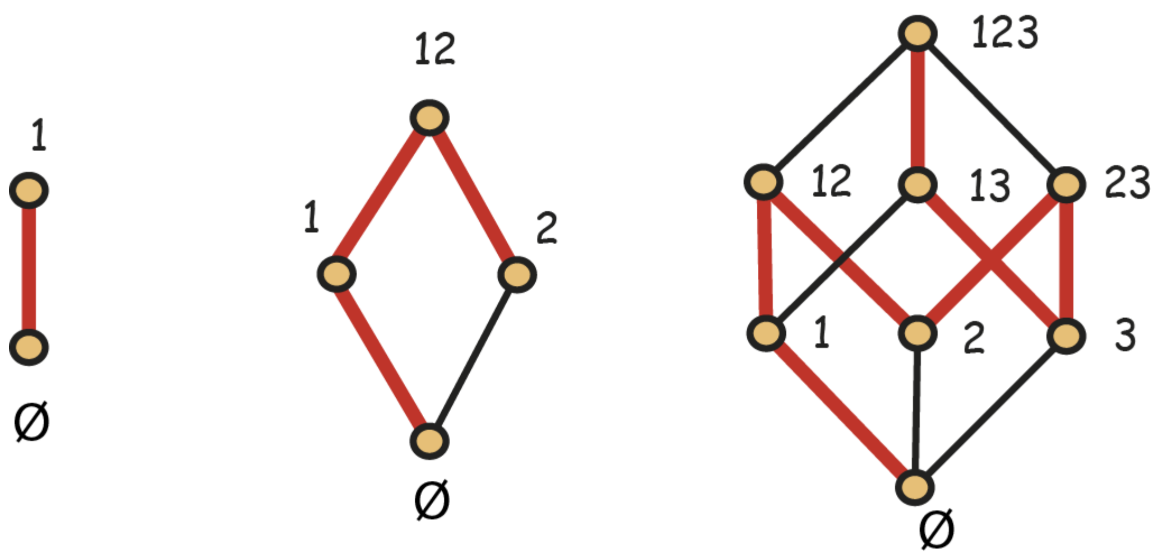
Question (Felsner and WTT, '95) Is it true that for every $t \geq 1$, there is a monotone hamiltonian path in the subset lattice $\mathbf{2}^t$?

Monotone Hamiltonian Paths

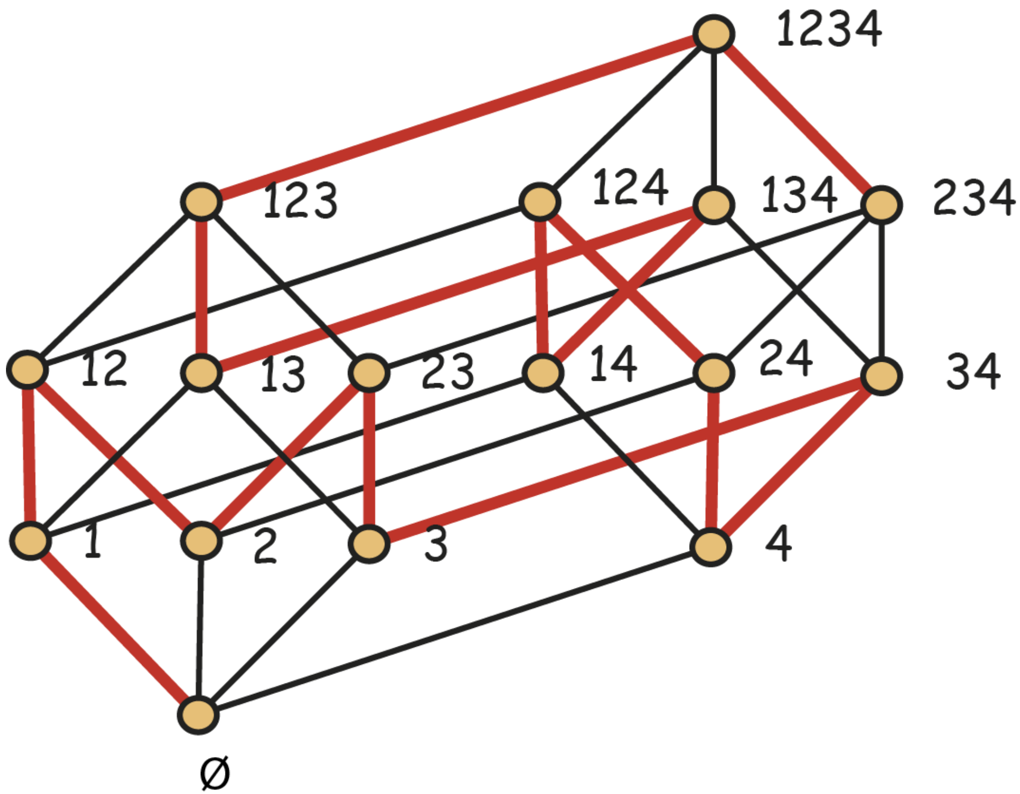
Definition Let $t \geq 1$ and set $n = 2^t$. A listing $(S_1, S_2, \ldots, S_n)$ of all subsets of $\{1, 2, \ldots, t\}$ is called a **monotone hamiltonian path** in the subset lattice $\mathbf{2}^t$ when

1. $|S_i \, \Delta \, S_{i+1}| = 1$
2. $S_1 = \emptyset$.
3. S_j contained in S_i implies $j \leq i+1$

Monotone Hamiltonian Paths



Monotone Hamiltonian Path in 2^4



Numerical Evidence – Mostly Positive

t	HP(t)
1	1
2	1
3	1
4	1
5	10
6	123
7	1, 492, 723
8	300, 000, 000, 000 (at least)
9	1 (at least)
10	1 (at least)

(4) Cartesian Products

Question How tight is the lower bound in the following inequality:

$$\max\{\dim(P), \dim(Q)\} \leq \dim(P \times Q) \leq \dim(P) + \dim(Q)$$

Theorem (WTT '85) $\dim(S_n \times S_n) = 2n - 2$.

(5) Construction Challenges

1. Construct explicitly a d-irreducible interval order for any $d \geq 4$.
2. Construct explicitly a poset P with $\dim(P) > 1 + \Delta(P)$.

(6) Forcing Standard Examples

Observation Let $f(n, d)$ be the least integer so that if P is a poset on n points and the dimension of P is at least $f(n, d)$, then P contains the standard example S_d.

Theorem (Füredi, Hajnal, Rödl and WTT, 91)
$f(n, 2) = (1 + o(1))\lg \lg n$

Theorem (Biró, Hamburger, and Pór, 14) For $d \geq 3$, $f(n, d) = o(n)$.

Theorem (Biró, Hamburger, Kierstead, Pór and WTT, 16+)
For $d \geq 3$,

$$f(n, d) \geq n^{1-(2d-1)/d(d-1)} / 8 \log n$$

(7) Standard Examples in Planar Posets

Question Does a planar poset with large dimension contain a large standard example? How about posets with planar cover graphs? How about posets whose cover graphs exclude a fixed minor?

Note Extensive work on connections between dimension and topological graph theory by Biró, Felsner, Joret, Howard, Keller, Li, Milans, Micek, Streib, WTT, Walczak, Wang, Wiechert and Young.