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On-line chain partitioning of semi-orders

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Abstract

We analyze the on-line chain partitioning of partially ordered sets as a two person game. One person builds an order one point at a time. The other person responds by making an irrevocable assignment of the new point to a chain of a chain partition. The value of the game val(w) is the least number of chains such that there is a strategy for the second player using no more than val(w) chains on orders of width w. The problem to estimate the function val(w) has been a challenge through last 25 years. Up to now the best known bounds are $\binom{w+1}{2} \leqslant \mathrm{val}(w) \leqslant \frac{5^w-1}{4}$. The up-growing variant, in which each point must be maximal at the moment of presentation, is done and the value of this variant is $\binom{w+1}{2}$. The chain partitioning game is also settled in the restriction for interval orders. The thesis is a comprehensive study of the on-line chain partitioning problem addressed to the class of semi-orders, i.e., interval orders that have a representation consisting of unit-length intervals. In particular, we prove the value of the game for semi-orders to be $2w-1$ and of the up-growing variant for semi-orders to be $\lfloor \frac{1+\sqrt{5}}{2} w \rfloor$. The latter reveals an astounding connection of the subject with the golden ratio. Also variants of the game in which incoming points are presented together with their interval representation are to be discussed.

Acknowledgments

The thesis is based on my research on partially ordered sets, in particular on on-line algorithms addressed to these structures, that began in 2003 during my M.Sc. studies. Paweł M. Idziak introduced me to the field, in general to combinatorics. I would like to thank him for stimulation, encouragement and many, many hours spent with me. My work would be much harder without the group of people, my friends, providing inspiration and great, creative atmosphere. I am especially indebted to Bartłomiej Bosek, Kamil Kloch, Tomasz Krawczyk and Grzegorz Matecki who worked with me through these years.

Contents

CHAPTER 1

Introduction

1.1. Outline

The central role in the thesis belongs to the on-line chain partitioning problem of partially ordered sets. In general, it touches the aspects of coloring problems through studying on-line algorithms. On-line approach to these classical combinatorial problems is getting more popular because of its natural application in many real-world problems.

An on-line algorithm can be understood as an algorithm working without knowledge of its entire input. Instead, step by step a part of the input is revealed and for each such part an irrevocable action must be taken before next part is introduced. This description surely applies in: real-time systems, routing in communication networks, task scheduling etc. In particular, consider the dynamic storage allocation problem. Each incoming task has starting time and finishing time. Each task has to get a memory slot in such a way that conflicting tasks (intersecting on the axis of time) have different slots. We are trying to minimize the number of slots used. Obviously, this can be modeled by the optimal coloring of the family of intervals in such a way that intersecting ones have different colors. In more general setting we are just looking for an optimal chain partition of a given interval order. But since tasks may come all the time we need in fact to construct on-line a partition of a given order, presented point by point, into chains.

On-line chain partitioning of a partially ordered set can be seen as a two-person game between Spoiler and Algorithm. The game is played in rounds. In each round Spoiler presents a new point enlarging already presented poset. Algorithm responds by making an irrevocable assignment to a chain in a constructed chain partition. Algorithm tries to minimize the number of chains used. Spoiler's goal is to spoil Algorithm's job by increasing the value $\mathrm{val}(w)$ describing the least number of chains such that there is a strategy for Algorithm using no more than $\mathrm{val}(w)$ chains on orders of width w. The problem to estimate the function $\mathrm{val}(w)$ has been a challenge through last 25 years. Up to now the best known estimation is $\binom{w+1}{2} \leqslant \mathrm{val}(w) \leqslant \frac{5^w-1}{4}$.

In the following we are going to address the on-line chain partitioning problem to the class of interval orders and its subclass of semi-orders. We establish the values of several variants of these games.

Chapter 1 provides a basic terminology and facts about partial orders. For a more detailed introduction to the field and a wider view on problems lying there the reader is sent to Trotter's book [**Tro92**] or a chapter in Handbook of Combinatorics [**GGL95**] devoted to partially ordered sets. Then comes a careful introduction to on-line approach for chain partitioning of posets. We present some ideas taken from the origins of research in on-line algorithms. Best known estimation for $\mathrm{val}(w)$ are to be discussed. We also discuss an up-growing variant of the game in which incoming points must be maximal at the moment of presentation. Finally in Section 1.4, we address on-line chain partitioning to interval orders. For a sample of research and a general look at the problems in the world of intervals we refer to Felsner's Ph.D. thesis [**Fel93**]. Exploring variants of the game in the interval realm we also consider games in which Spoiler presents intervals instead of points. This chapter contains already known results but sometimes we give arguments simply to introduce all necessary notions and show how they work.

The rest of the thesis is a comprehensive study of the on-line chain partitioning problem addressed to the class of semi-orders, i.e., interval orders that have a representation consisting of unit-length intervals.

Chapter 2 introduces the class of semi-orders and is a general overview of results to be presented in Chapter 3 and 4.

In Chapter 3 we deal with the variants of the game in which Spoiler presents points of a semi-order without their interval representation. First we relatively easily establish the value of non up-growing variant to be $2w - 1$ for semi-orders of width w. Then a surprising connection between the value of the up-growing variant for semi-orders and the golden ratio is to be presented (Section 3.2). It will turn out that the value in this variant is $\lfloor \frac{1+\sqrt{5}}{2} w \rfloor$.

Chapter 4 is devoted to the variant in which Spoiler presents unit-length intervals. Here remains the last gap left in the area. How many colors suffice to color incoming unit-length intervals in such a way that intersecting ones receive different colors? The best bounds in terms of the largest clique of presented collection of intervals as well as some arguments which directions should not be followed will be shown.

1.2. Partially ordered sets – basic concepts

A partially ordered set is a pair (X, R), where X is a set (commonly set of points) and R is a binary relation on X such that:

(i) xRx, (R is reflexive)
(ii) xRy implies $\neg yRx$, (R is antisymmetric)
(iii) xRy and yRz imply xRz. (R is transitive)

for all x, y, z in X. We often abbreviate "partially ordered set" as either order or poset. An order is finite (countable) if it has finite (countable) number of points. Almost in all our considerations we deal with finite posets.

The commonly used notation for a partially ordered set $\mathbf{P}$ is $\mathbf{P} = (P, \leqslant)$. We then write $x < y$ (or $y > x$) in $\mathbf{P}$ to mean $x \leqslant y$ and $x \neq y$. For $X, Y \subseteq P$ by $X \leqslant Y$ we mean $x \leqslant y$ in $\mathbf{P}$ for all $x \in X$, $y \in Y$. We also write simply $X < y$ when $Y = \{y\}$.

Two points x, y are said to be comparable in $\mathbf{P}$, denoted $x \sim y$, if $x \leqslant y$ or $x \geqslant y$. Otherwise, we say x and y are incomparable in $\mathbf{P}$ and write $x \parallel y$.

Partially ordered set may be represented graphically by a digraph. We may omit arrows between points connected by a sequence of arrows (this relation is clear from transitivity). Moreover, drawing x lower in the plane than y whenever $x < y$ allows us to drop arrowheads. Such a representation is called a diagram (or Hasse diagram, see Figure 1.1 for an example).

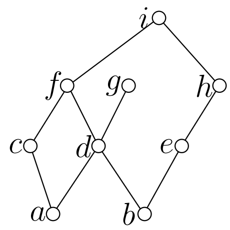

FIGURE 1.1. Diagram of poset $\mathbf{P} = (\{a, \ldots, i\}, \leqslant)$ with $a < c$, $\{a, b\} < d$, $b < e$, $\{a, b, c, d\} < f$, $\{a, b, d\} < g$, $\{b, e\} < h$, $\{a, b, c, d, e, f, h\} < i$.

Let $\mathbf{P} = (P, \leqslant)$ be a partially ordered set. A subset of P is a chain in $\mathbf{P}$ if every two elements in the subset are comparable. Throughout the text we sometimes use $\mathbf{n}$ to denote a poset formed by a n-element chain. Dually, a subset of P is an antichain in $\mathbf{P}$ if every two distinct elements in the subset are incomparable. The height of $\mathbf{P}$, denoted height($\mathbf{P}$), is the maximal cardinality of a chain in $\mathbf{P}$, and the width of $\mathbf{P}$, denoted width($\mathbf{P}$), is the maximal cardinality of an antichain in $\mathbf{P}$.

By the upset of $X \subseteq P$ in $\mathbf{P} = (P, \leqslant)$ we mean

$$X{\uparrow}_{\mathbf{P}} = \{p \in P : \text{ there is } x \in X \text{ such that } x < p\}.$$

Dually we define the downset $X{\downarrow}_{\mathbf{P}}$ of X in $\mathbf{P}$:

$$X{\downarrow}_{\mathbf{P}} = \{p \in P : \text{ there is } x \in X \text{ such that } x > p\}.$$

If $X = \{x\}$, we prefer to write $x{\uparrow}_{\mathbf{P}}$ instead of $\{x\}{\uparrow}_{\mathbf{P}}$. The reference to the poset is often omitted when it is clear from the context.

An element $x \in P$ is maximal (minimal) in $\mathbf{P}$ if there is no $y \in P$ for which $x < y$ in $\mathbf{P}$ ($y < x$ in $\mathbf{P}$). The set of maximal (minimal) points of $\mathbf{P}$ is denoted by max($\mathbf{P}$) (min($\mathbf{P}$)). For a finite chain C in $\mathbf{P}$ by a top(C) we mean the unique, maximal point of C.

By $\mathbf{P} + \mathbf{Q}$ we denote the disjoint sum of posets $\mathbf{P}$ and $\mathbf{Q}$, obtained by taking disjoint copies of the two posets with no comparabilities between them. By $\mathbf{P} \bowtie \mathbf{Q}$ we denote the serial sum of posets $\mathbf{P}$ and $\mathbf{Q}$, obtained by putting a copy of $\mathbf{Q}$ completely above a copy of $\mathbf{P}$, i.e., all points from $\mathbf{P}$ are below all points from $\mathbf{Q}$ in $\mathbf{P} \bowtie \mathbf{Q}$.

One of the fundamental results in the area, which played important role in motivating research in posets, is Dilworth's decomposition theorem. There are several elementary proofs; the presented one is given by Perles (1963).

THEOREM 1.1 (Dilworth [**Dil50**]). *If* $\mathbf{P} = (P, \leqslant)$ *is a finite poset of width* w *then there exists a partition* $P = C_1 \cup \ldots \cup C_w$ *where each* C_i *is a chain.*

PROOF. We use induction on $|P|$. Choose $x \in \min(\mathbf{P})$ and $y \in \max(\mathbf{P})$ such that $x \leqslant y$ and let $\mathbf{Q} = (P - \{x, y\}, \leqslant)$. If $\text{width}(\mathbf{Q})$ is less than w, we may inductively cover $\mathbf{Q}$ with at most $w - 1$ chains and pick $\{x, y\}$ as w-th chain to cover the whole $\mathbf{P}$. Otherwise, we have $\text{width}(\mathbf{Q}) = w$, therefore $x < y$. Choose A, any w-element antichain in $\mathbf{Q}$. Observe that $x \in A{\downarrow_{\mathbf{P}}}$ and $y \in A{\uparrow_{\mathbf{P}}}$.

Since the posets formed by $A \cup A{\downarrow_{\mathbf{P}}}$ and $A \cup A{\uparrow_{\mathbf{P}}}$ are smaller than $\mathbf{P}$ they can be inductively covered with w chains $D_1, \ldots, D_w$ and $U_1, \ldots, U_w$, respectively. After renumbering we may assume that $\emptyset \neq D_i \cap U_i \subseteq A$. Now, the chains $D_1 \cup U_1, \ldots, D_w \cup U_w$ cover our poset $\mathbf{P}$. □

1.3. On-line approach to chain partitioning

The origins of research in on-line algorithms lie in recursive combinatorics. This was a program (in the 70's and the 80's) to extend the domain of finite combinatorics to recursive objects.

One of the attractions of finite combinatorics is the explicit descriptions of the objects under consideration. These explicit descriptions are usually lost when one passes to infinite combinatorics. For example, we propose the countable version of Dilworth's theorem (in fact it can be easily proved for posets of any cardinality using compactness theorem).

THEOREM 1.2. *If* $\mathbf{P} = (P, \leqslant)$ *is a countable poset of width* w *then there exists a partition* $P = C_1 \cup \ldots \cup C_w$ *where each* C_i *is a chain.*

PROOF. Let $P = \{p_i\}_{i \in \mathbb{N}}$. We show by an induction that for all $i \in \mathbb{N}$ there exist chains $C_1^i, \ldots, C_w^i$ such that

(i) $C_1^i \cup \ldots \cup C_w^i = \{p_0, \ldots, p_i\}$,
(ii) $C_k^j \subseteq C_k^i$, for all $j < i$ and $1 \leqslant k \leqslant w$,
(iii) if $\mathbf{Q}$ is a finite subposet of $\mathbf{P}$ then $C_1^i, \ldots, C_w^i$ can be extended to chains that cover $\mathbf{Q}$.

This will give a partition of $\mathbf{P}$ into chains: $\bigcup_{i \in \mathbb{N}} C_1^i, \ldots, \bigcup_{i \in \mathbb{N}} C_w^i$.

Put $C^0_k = \emptyset$ for all k but $k = 0$ for which $C^0_0 = \{p_0\}$. Then all invariants are satisfied: in particular (iii) follows from Dilworth's theorem. Now, suppose that we have $C^i_1, \ldots, C^i_w$ satisfying (i)–(iii). We prove that there is m $(1 \leqslant m \leqslant w)$ such that (i)–(iii) hold also for $C^{i+1}_1, \ldots, C^{i+1}_w$ where

$$C^{i+1}_k = C^i_k, \text{ for } k \neq m,$$
$$C^{i+1}_m = C^i_m \cup \{p_i\}.$$

Obviously, (i) and (ii) hold. If for all m's (iii) fails then for each one we have finite poset $\mathbf{Q}_m \subseteq \mathbf{P}$ witnessing that failure. Now, $\mathbf{Q} = \mathbf{Q}_1 \cup \ldots \cup \mathbf{Q}_w$ is also finite and it contradicts (iii) for $C^i_1, \ldots, C^i_w$ which ends the proof. □

While the above argument shows that there is a cover of $\mathbf{P}$ using w chains it does not show how to effectively or computationally produce that cover. The problem is that at each stage i, in order to decide which chain assign to p_i we must perform the impossible task of considering each of the infinitely many finite subposets of $\mathbf{P}$.

Although recursive objects may be infinite, they are described by the algorithm associated with them. While we cannot hope to assimilate the total amount of information about an infinite structure that is in some sense stored in the finite algorithm, we certainly have complete information about any finite part of the structure. The support for the thesis that recursive combinatorics is a natural extension of finite combinatorics comes from the experience that the arguments used to prove theorems have the flavor and style of argument used in finite combinatorics. One can find contributions to this area in [**Bea76, Sch80, Sch82, Kie81b, Kie83**] studying recursive graphs, highly recursive graphs, their recursive chromatic number etc. and in [**Kie81a, KT81, KMT84**] on recursive partial orders.

To deal with these problems researchers often used the same tool – a special two person game. In this game, first player builds, point by point, considered structure: it could be a graph, an order etc. The second player constructs desired combinatorial structure, e.g. matching, coloring, chain partitioning. He is forced to make some irrevocable decisions without the knowledge of future points. Quickly, it turned out that such games, called on-line, are interesting in their own. In the following we focus on the analysis of on-line chain partitioning games.

On-line chain partitioning is a two person game. We call the players: Spoiler and Algorithm. The game is played in rounds. During each round Spoiler introduces a new point and describes comparabilities between the new point and points already presented. Algorithm responds by making an irrevocable assignment of the new point to one of the chains in the chain partition. We usually refer to Algorithm's chains as colors and consequently to obtained chain partitioning as coloring.

Suppose that Spoiler has already presented poset $\mathbf{P} = (P, \leqslant)$. Then by the presentation sequence $P^<$ we mean a linear ordering of P agreeing with the order in which Spoiler presented points of $\mathbf{P}$. For an Algorithm's strategy (on-line algorithm) A, a partial order $\mathbf{P} = (P, \leqslant)$ with a presentation sequence $P^<$ and a new, just presented, point x enlarging $\mathbf{P}$ let $A(\mathbf{P}; P^<)(x)$ be the chain (color) used by A on x in this setting.

Fact 1.3. For the on-line chain partitioning game on orders of width $w \geqslant 1$ define:

(i) $\mathrm{val}_{\mathrm{S}}(w)$ to be the largest integer n for which Spoiler has a strategy that forces any Algorithm to use n chains,
(ii) $\mathrm{val}_{\mathrm{A}}(w)$ to be the least integer n such that Algorithm has a strategy using at most n chains.

Then $\mathrm{val}_{\mathrm{S}}(w) = \mathrm{val}_{\mathrm{A}}(w)$.

Proof. Since the best Spoiler's strategy witnessing $\mathrm{val}_{\mathrm{S}}(w)$ forces this number of chains, in particular, against the best Algorithm's strategy we get $\mathrm{val}_{\mathrm{S}}(w) \leqslant \mathrm{val}_{\mathrm{A}}(w)$.

To achieve the converse inequality we enumerate all Algorithm's strategies in a sequence $\{A_i\}_{i\in\mathbb{N}}$. By the definition of $\mathrm{val}_{\mathrm{A}}(w)$ each A_i can be forced to use $\mathrm{val}_{\mathrm{A}}(w)$ chains, say with $\mathbf{Q}_i$ with presentation sequence $Q_i^<$.

Observe that during the game Spoiler may, in some sense, reset the game forgetting about all already presented points and introducing new ones completely above them. This context left below does not restrict Spoiler (new points will not contribute to the width with old ones), i.e., Spoiler may perform any of his strategies. For $i \in \mathbb{N}$ and an on-line poset $\mathbf{P}$ with presentation sequence $P^<$ let $(i, \mathbf{P})$ be the index of an on-line algorithm $A_{(i,\mathbf{P})}$ such that

$$A_{(i,\mathbf{P})}(\mathbf{Q}; Q^<)(x) = A_i(\mathbf{P} \bowtie \mathbf{Q}; P^<, Q^<)(x),$$

where point x is assumed in $\mathbf{P} \bowtie \mathbf{Q}$ to be presented completely above $\mathbf{P}$. Loosely speaking $A_{(i,\mathbf{P})}$ is the behavior of A_i after $\mathbf{P}$ is introduced and fixed to be at the bottom of the whole incoming poset.

Now, we present a new strategy for Spoiler which will force all A_i's to use at least $\mathrm{val}_{\mathrm{A}}(w)$ chains and therefore $\mathrm{val}_{\mathrm{S}}(w) \geqslant \mathrm{val}_{\mathrm{A}}(w)$. This strategy looks as follows:

(i) Present $\mathbf{P}_0 = \mathbf{Q}_0$ forcing A_0 to use at least $\mathrm{val}_{\mathrm{A}}(w)$ chains.
(ii) Given $\mathbf{P}_{i-1}$, for $i \geqslant 1$, with its presentation sequence $P_{i-1}^<$ enlarge $\mathbf{P}_{i-1}$ to $\mathbf{P}_i = \mathbf{P}_{i-1} \bowtie \mathbf{Q}_{(i,\mathbf{P}_{i-1})}$ with the presentation $P_{i-1}^<, Q_{(i,\mathbf{P}_{i-1})}^<$.

Note that $\mathbf{P}_i$ is constructed to force $A_0, \dots, A_i$ to use at least $\mathrm{val}_{\mathrm{A}}(w)$ colors. $\square$

As $\mathrm{val_S}(w)$ and $\mathrm{val_A}(w)$ is the same we simply call it the value of the game for orders of width w, denoted further by $\mathrm{val}(w)$. By Dilworth's theorem Algorithm will be forced to use at least w chains on order of width w, i.e., $\mathrm{val}(w) \geqslant w$.

The best known results estimating $\mathrm{val}(w)$ were done in early 80's. A construction of Szemerédi (published by Kierstead in [**Kie86**]) proves lower bound. Kierstead presented a strategy for Algorithm using exponential, but bounded (in terms of w), number of chains.

THEOREM 1.4 (Szemerédi [**Kie86**]; Kierstead [**Kie81a**]).

$$\binom{w+1}{2} \leqslant \mathrm{val}(w) \leqslant \frac{5^w - 1}{4}.$$

This is already a complicated result, and no progress has been made on the general problem for the last 25 years. Kierstead in [**Kie81a**] presented also a lower bound $4w - 3 \leqslant \mathrm{val}(w)$, better for first few values of w. The precise value of $\mathrm{val}(w)$ is only known for $w \leqslant 2$, where $\mathrm{val}(2) = 5$ (by Kierstead's lower bound and Felsner's upper bound given in [**Fel97**]).

THEOREM 1.5 (Kierstead [**Kie81a**]; Felsner [**Fel97**]).

$$\mathrm{val}(2) = 5.$$

There is still a gap for $w = 3$. Recently, Bosek improved the best known upper bound from 31 to 16.

THEOREM 1.6 (Kierstead [**Kie81a**]; Bosek [**Bos08**]).

$$9 \leqslant \mathrm{val}(3) \leqslant 16.$$

The question is: whether $\mathrm{val}(w)$ is bounded by a polynomial in w? Since this still remains challenging some restricted variants are analyzed.

EXAMPLE 1.7. Bounding h, the height of presented order, in particular we restrict Spoiler to introduce at most $h \cdot w$ points. For $h \leqslant 2$, the problem is essentially equivalent to on-line matching in bipartite graph and as we will see the value of this game is $\lfloor \frac{3}{2} w \rfloor$.

The strategy for Spoiler forcing $\lfloor \frac{3}{2} w \rfloor$ colors (chains) is simple. First, Spoiler introduces an antichain A of size w. Algorithm has to use w colors. Now, Spoiler presents an antichain B_0 above A (i.e., $a < b$ for all $a \in A$, $b \in B_0$) of size $\lfloor \frac{w}{2} \rfloor$. Algorithm coloring B_0 may use colors from A or brand new colors. Let $A_0 \subseteq A$ be the set of points which colors has been used on B_0. Then Spoiler presents an antichain B_1 of size $|A_0|$ (if A_0 is empty no points are presented) above A_0 and incomparable with all other points. Algorithm is forced to use new colors on points from B_1. Summing up $|A| + (|B_0| - |A_0|) + |B_1| = \lfloor \frac{3}{2} w \rfloor$ colors are used. Obviously, the Spoiler's construction has width w.

On the other hand, greedy Algorithm, i.e., Algorithm coloring with a new color only if it has to, uses no more than $\lfloor \frac{3}{2}w \rfloor$ colors on orders of width at most w and height 2. Such an order may have at most $2w$ points. All colors (chains) used consist of one or two points. In the worst case the number of colors used only once is maximized. But the greedy property guarantees that points colored with a unique color (used once) must form an antichain and therefore there may be at most w such colors. Thus, in the worst case for the even w's we have w unique colors and at most $\frac{2w-w}{2}$ colors being doubletons and for the odd w's we have $w-1$ and $\frac{2w-(w-1)}{2}$, respectively. In both cases it gives at most $\lfloor \frac{3}{2}w \rfloor$ colors. □

Felsner [**Fel97**] introduced a variant of the chain partitioning problem. He restricts possible inputs (Spoiler moves) by the rule that the sequence in which elements are released is a linear extension of the poset, i.e., a comparability of a new element x to the former y's has to be of the form $y < x$. In other words, each new point is maximal at the moment of its arrival. On-line posets with this property are called up-growing. Felsner settled the precise value of this variant.

THEOREM 1.8 (Felsner [**Fel97**]). *The value of the on-line chain partitioning game for up-growing orders of width w is $\binom{w+1}{2}$.*

Another natural restriction is to limit Spoiler to present orders from some special class. In particular, the classes of interval orders and semiorders are of our interest.

1.4. On-line chain partitioning of interval orders

An order $\mathbf{P} = (P, \leqslant)$ is an interval order if there is a function I assigning to each element $x \in P$ a closed interval $I(x) = [l_x, r_x]$ of the real line $\mathbb{R}$ so that for all $x, y \in P$ we have $x < y$ in $\mathbf{P}$ iff $r_x < l_y$. Such a function I is called the representation of $\mathbf{P}$. See Figure 1.2 for an example.

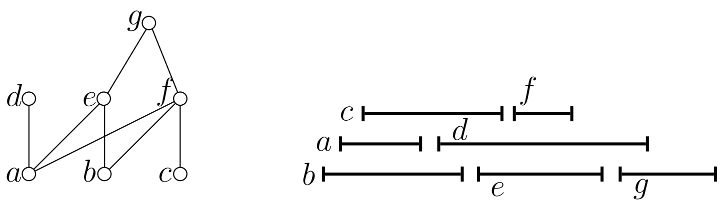

FIGURE 1.2. Interval order $\mathbf{P} = (\{a, b, c, d, e, f, g\}, \leqslant)$ with its representation.

The value of the on-line chain partitioning game for interval orders was settled in the early 80's by Kierstead and Trotter. They formulated the result in terms of recursive combinatorics. Several years later

Chrobak and Ślusarek proved the same but this time in terms of on-line algorithms.

THEOREM 1.9 (Kierstead, Trotter [**KT81**]; Chrobak, Ślusarek [**CS88**]). *The value of the on-line chain partitioning game for interval orders of width* w *is* $3w - 2$.

There is one more, subtle difference between these two results. Playing the on-line chain partitioning game Spoiler presents, one by one, points of order. In the case of interval order, it is maybe even more natural to present representation, i.e., intervals. Then Algorithm would have a task to color these intervals in such a way that intersecting ones have different colors. The corresponding notion for the width, in this variant, is the clique-size – the maximal size of the set of mutually intersecting intervals. Standard variant, in which Spoiler presents points is called a variant without representation and the one in which Spoiler presents intervals is a variant with representation. Below we provide an example showing that these two settings are not equivalent.

EXAMPLE 1.10. Let $\mathbf{P} = (\{a, b, c\}, \leqslant)$ be the interval order presented by Spoiler (see Figure 1.3). Consider game with points (on the left) and with representation (on the right). Numbers in the figure indicates colors (chains) assigned by Algorithm. In the setting without representation Spoiler may introduce point dominating a and no other points. A moment of thought reveals that it is impossible in the case of intervals.

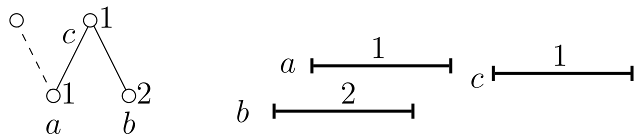

FIGURE 1.3. Points against intervals

Surprisingly, it turns out that the values of these two considered variants are the same. Although argument is more or less the same Kierstead and Trotter analyzed the game without representation while Chrobak and Ślusarek did it for intervals. For the sake of presentation we provide the argument on intervals but it is easy to see that proposed strategy for Algorithm can handle without representation.

PROOF OF THEOREM 1.9. First, we provide the strategy $S(w)$ for Spoiler forcing Algorithm to use at least $3w - 2$ colors (chains) on the collection of intervals (interval order) of clique-size (width) at most w. The strategy $S(1)$ is trivial, it suffices to present a single interval. To construct $S(k + 1)$ present first many copies of $S(k)$ placed in disjoint pieces of the real line. Inductively, Algorithm on each copy uses $3k - 2$ colors (if more we forget about the surplus). If $3k + 3$ or more colors are used in general, we are done. Otherwise, Algorithm has only $\binom{3k+2}{3k-2}$ options to color each copy of $S(k)$ and presenting appropriate number of them Spoiler forces four copies, say C_1, C_2, C_3, C_4 in order from left to right, with the same set of $3k-2$ colors. Now, Spoiler introduces two intervals: the first interval covers all intervals from C_1 and is disjoint with all the rest, the second analogously covers C_4 (see Figure 1.4). Of course, Algorithm has to use new color (comparing with the fixed

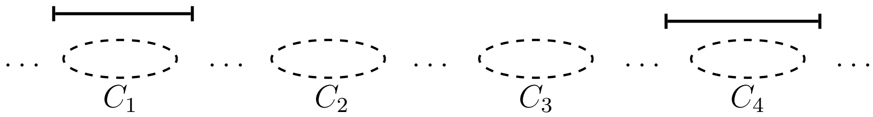

FIGURE 1.4. Building $S(k+1)$: Place of two new intervals intersecting C_1 and C_4

set of $3k-2$ colors). If both intervals are colored with the same color then Spoiler introduces next two intervals as in Figure 1.5. Otherwise,

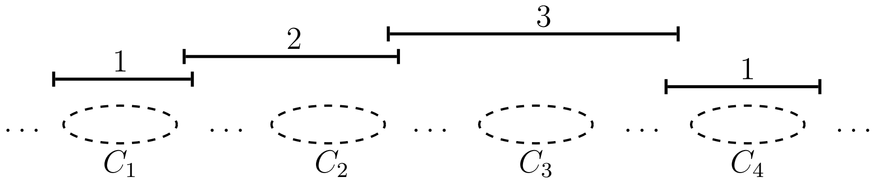

FIGURE 1.5. Building $S(k+1)$: Algorithm used the same new color

if Algorithm uses two different colors then the third color is forced by presenting an interval as shown in Figure 1.6. It is clear that the clique-

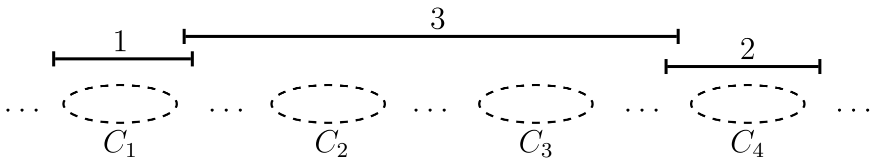

FIGURE 1.6. Building $S(k+1)$: Algorithm used two different colors

size of the presented collection (in all cases) is at most $k+1$ and since Spoiler forced at least $(3k-2)+3$ colors we are done.

In order to prove upper bound we present a strategy $A(w)$ for Algorithm using at most $3w-2$ colors on any collection of clique-size at most w. Again, $A(1)$ is trivial. The strategy $A(k+1)$ maintains a partition of presented intervals into two sets G and R such that the clique-size of G is at most k. When Spoiler introduces a new interval x the strategy $A(k+1)$ puts it into G if it does not violate the clique-size condition for $G \cup \{x\}$. Otherwise, x is put into R. To deal with intervals in G, $A(k+1)$ recursively calls $A(k)$. Therefore at most $3k-2$ colors are used on G. It suffices to show that intervals in R may be on-line colored using 3 colors.

Observe, that each interval $r \in R$ must form a clique of size $k+1$ together with some k intervals $g_1, \ldots, g_k$ from G. Moreover, for $r' \in R - \{r\}$ the intersection of r' with the non-empty interval $g_1 \cap \ldots \cap g_k \cap r$ has to be empty as otherwise there would be a clique of size $k+2$ (see Figure 1.7). Hence, no interval from R is covered by the sum of all other intervals from R. Using that we will argue that interval from R

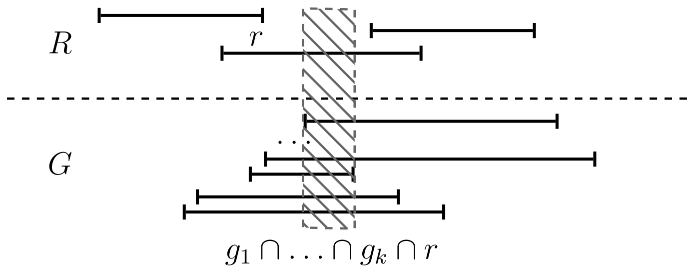

FIGURE 1.7. $r' \cap (g_1 \cap \ldots \cap g_k \cap r) = \emptyset$ for all $r' \in R - \{r\}$

may intersect with at most two other intervals from R and so a greedy algorithm uses at most 3 colors on R. Indeed, no $r \in R$ contains other interval from R so all intersections $r \cap r'$ with $r' \in R - \{r\}$ must contain an endpoint of r. This implies that if r intersects more than two intervals from R at least two such intervals contain one of its endpoints, say the left endpoint (see Figure 1.8). But then one of these

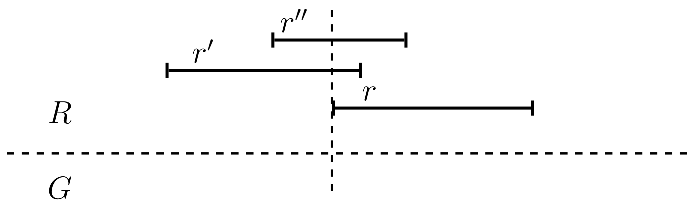

FIGURE 1.8. r' and r'' contain the left end of r and r'' is covered by the sum of r and r'

two intervals is contained in the sum of two others. This contradiction shows that $A(k+1)$ colors greedily intervals from R with at most three colors. □

Recently, the values of up-growing variants of the on-line chain partitioning game for interval orders have been established. For the proof of the case when Spoiler presents points (i.e., without representation) we send the reader to [**BBM07**].

THEOREM 1.11 (Baier, Bosek, P.M. [**BBM07**]). *The value of the on-line chain partitioning game for up-growing interval orders of width w without representation is $2w - 1$.*

The up-growing case with representation appears to be really relaxing for Algorithm. In this setting he has enough information to use optimal (off-line) number of colors. In other words Spoiler has no tools to cheat the Algorithm, in fact the nearest-fit algorithm.

THEOREM 1.12 (Broniek [**Bro05**]). *The value of the on-line chain partitioning game for up-growing interval orders of width w presented with representation is w.*

PROOF. The strategy for Algorithm is as simple as follows: from all legal colors (i.e., not used for intervals intersecting the incoming interval) choose one used rightmost (or more precisely one of the legal colors used for intervals with the right endpoints nearest to the incoming interval). We prove that this strategy (called nearest-fit algorithm) uses no more colors than the clique-size of presented collection of intervals. In order to get contradiction suppose that Spoiler forced nearest-fit algorithm to use $w + 1$ colors on the collection of clique-size at most w. Let x be the first interval colored with $(w + 1)$-th color. By $\text{top}(\gamma)$, the top of the color γ, we mean the rightmost interval colored with γ. Let Tops be the set of tops of the colors at the moment when x is presented. For each already used color γ, the interval x must intersect an interval colored with γ (otherwise, the nearest-fit algorithm would use one of the old colors on x). By the up-growing property we get that x intersects all intervals from Tops. Consider interval $y = \text{top}(\alpha) \in \text{Tops}$ – the one with leftmost right endpoint among intervals from Tops.

We are going to prove that each color γ is used for an interval containing the right endpoint of y and therefore these intervals together with x and y form a clique of size $w + 1$. For any $\beta \neq \alpha$ let $t = \text{top}(\beta) \in \text{Tops}$. If t contains right endpoint of y we have nothing to prove. Otherwise, let t_0 be the leftmost interval colored with β and completely to the right of y (see Figure 1.9). Now, consider the situation at the moment when t_0 arrives. Again, by the up-growing property y is already presented (i.e., before t_0). Since the nearest-fit algorithm gives t_0 color β (but not α) there must be an interval colored with β, with right endpoint to the right of the right endpoint of y. On the other hand, t_0 was the leftmost interval colored with β and completely to the right of y. All of this implies that there must be an interval colored with β containing the right endpoint of y (see Figure 1.9).

We have just argued that intervals containing the right endpoint of y form a clique of size $w + 1$ contradicting our assumption. □

The results in this section establish the values of on-line chain partitioning games for interval orders in all four considered variants.

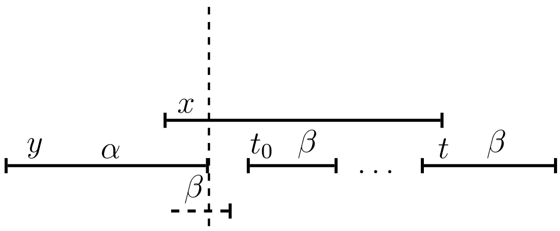

FIGURE 1.9. There must be an interval colored with β intersecting our vertical line

In the following we investigate the same variants of the game addressed to the class of semi-orders, i.e., interval orders with a unit-length representation.

CHAPTER 2

Semi-orders – an overview

An order $\mathbf{P} = (P, \leqslant)$ is a semi-order if there is a function I assigning to each point $x \in P$ a closed, unit-length interval $I(x) = [l_x, l_x + 1]$ of the real line $\mathbb{R}$ so that for all $x, y \in P$ we have $x < y$ in $\mathbf{P}$ iff $l_x + 1 < l_y$. In other words, an interval order is a semi-order if it has a representation formed by unit-length intervals. By a straightforward argument one can locally rescale intervals one by one to show that $\mathbf{P}$ is a semi-order iff $\mathbf{P}$ has a proper representation, i.e., representation in which no interval is in the interior of other interval.

Before studying variants of on-line chain partitioning for semi-orders we recall some useful characterization theorems.

Theorem 2.1 (Fishburn [**Fis70**]). *Let* $\mathbf{P} = (P, \leqslant)$ *be a poset. Then the following statements are equivalent:*

(i) $\mathbf{P}$ *is an interval order,*
(ii) $\mathbf{P}$ *is a* $(\mathbf{2}+\mathbf{2})$*-free poset, i.e.,* $\mathbf{P}$ *does not contain elements* $a, b, c, d \in P$ *such that:* $a < b$, $c < d$, $a \parallel d$ *and* $c \parallel b$ *(see Figure 2.1),*
(iii) *for* $x, y \in P$ *we have* $x{\uparrow} \subseteq y{\uparrow}$ *or* $y{\uparrow} \subseteq x{\uparrow}$,
(iv) *for* $x, y \in P$ *we have* $x{\downarrow} \subseteq y{\downarrow}$ *or* $y{\downarrow} \subseteq x{\downarrow}$.

□

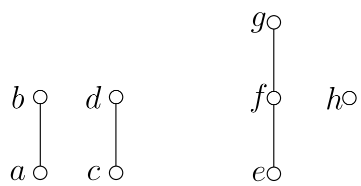

Figure 2.1. $(\mathbf{2}+\mathbf{2})$ and $(\mathbf{3}+\mathbf{1})$ posets

The ordering of downsets (upsets) corresponds to the order of left (right) endpoints in an interval representation. These orderings can be used to define extensions of $\mathbf{P}$ as follows:

$$p \prec_{\downarrow} q \text{ iff } p{\downarrow} \subsetneq q{\downarrow} \text{ or } (p{\downarrow} = q{\downarrow} \text{ and } p{\uparrow} \supsetneq q{\uparrow}),$$

$$p \prec_{\uparrow} q \text{ iff } p{\uparrow} \supsetneq q{\uparrow} \text{ or } (p{\uparrow} = q{\uparrow} \text{ and } p{\downarrow} \subsetneq q{\downarrow}).$$

In general, two orderings $\prec_{\downarrow}$ and $\prec_{\uparrow}$ need not to coincide. Indeed, in a $(\mathbf{3}+\mathbf{1})$ poset from Figure 2.1 we have $h \prec_{\downarrow} f$ whereas $f \prec_{\uparrow} h$. Here comes characterization theorem for semi-orders, recalled from Trotter's book [**Tro92**] and combined with the result of Scott and Suppes.

THEOREM 2.2 (Scott, Suppes [**SS58**]). *Let* $\mathbf{P} = (P, \leqslant)$ *be an interval order. Then the following statements are equivalent.*

(i) $\mathbf{P}$ *is a semi-order,*
(ii) $\mathbf{P}$ *is a* $(\mathbf{3}+\mathbf{1})$*-free order, i.e.,* $\mathbf{P}$ *does not contain elements* $e, f, g, h \in P$ *such that* $e < f < g$ *and* $h \parallel \{e, f, g\}$ *(see Figure 2.1),*
(iii) *The two orderings* $\prec_{\downarrow}$ *and* $\prec_{\uparrow}$ *are identical in* $\mathbf{P}$*.*

□

Let $\mathbf{P} = (P, \leqslant)$ be a semi-order and let $p, q \in P$. Theorem 2.2 justifies the following definition

(1) $$p \prec q \text{ iff } p{\downarrow} \subsetneq q{\downarrow} \text{ or } p{\uparrow} \supsetneq q{\uparrow}.$$

The order $(P, \prec)$ is an extension of $\mathbf{P}$ (in fact irreflexive extension) formed by a serial sum of sets of points with the same up- and downsets.

In Chapter 1 we discussed variants of on-line chain partitioning games for interval orders. Their values are summed up in Table 2.1.

up-growing	**representation**		
-	-	$3w-2$	Kierstead, Trotter
-	+	$3w-2$	Chrobak, Ślusarek
+	-	$2w-1$	Baier, Bosek, P.M.
+	+	w	Broniek

TABLE 2.1. The values of on-line chain partitioning games for **interval orders**

The main goal of the thesis is to establish the values of the analogous games for semi-orders (see Table 2.2). Variants without representation, in which Spoiler presents points of semi-order, are considered in Chapter 3. The general (not necessarily up-growing) case turns out to be relatively easy especially comparing with up-growing case where surprisingly golden ratio comes into the play. A proof with detailed description of our technique is to be presented. Variants of the game in which Spoiler presents representation are still not completely settled. While there is nothing to do in up-growing variant (by Theorem 1.12 and off-line lower bound) the non up-growing case seems to be challenging. Moreover, it can make a difference presenting proper intervals instead of unit-length intervals as in the latter Algorithm has more information. All of this will be discussed in Chapter 4.

Before proving presented bounds for consecutive variants we start with the following easy fact, helping setting them off.

up-growing	**representation**	
-	-	$2w-1$
-	+	$\lfloor \frac{3}{2}w \rfloor \leqslant \cdots \leqslant 2w-1$
+	-	$\lfloor \frac{1+\sqrt{5}}{2}w \rfloor$
+	+	w

TABLE 2.2. The values of on-line chain partitioning games for **semi-orders**

FACT 2.3. The values of all four considered variants of on-line chain partitioning games for semi-orders are between w and $2w-1$.

PROOF. It is clear that Spoiler has the weakest position in the up-growing variant with representation. Thus, the value of this variant bounds from below the other three. On the other hand, the most difficult variant for Algorithm is non up-growing, without representation and its value bounds from above the rest.

The lower bound w is an obvious off-line bound. To force w colors Spoiler may just introduce a clique of w unit-length intervals.

To prove the upper bound we show that a greedy strategy for Algorithm (in non up-growing variant without representation) never needs more than $2w-1$ colors. Let x be the point just presented by Spoiler and $\mathrm{Inc}(x)$ the set of points incomparable with x. The only colors forbidden to use on x are those in $\mathrm{Inc}(x)$. Clearly $\mathrm{width}(\mathrm{Inc}(x)) \leqslant w-1$ since the width of the whole order does not exceed w. Moreover, $\mathrm{height}(\mathrm{Inc}(x)) \leqslant 2$ as the presented order is $(\mathbf{3}+\mathbf{1})$-free (see Theorem 2.2). This implies that $|\,\mathrm{Inc}(x)| \leqslant 2(w-1) = 2w-2$ (see Figure 2.2), proving that at least one of $2w-1$ colors is legal for x. $\square$

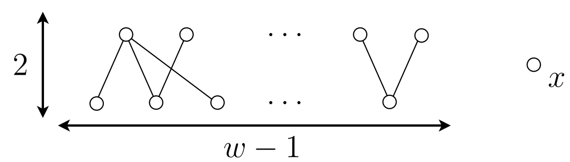

FIGURE 2.2. $|\mathrm{Inc}(x)| \leqslant 2w-2$

CHAPTER 3

Semi-orders without representation

We are going to consider games in which Spoiler presents points of a semi-order without their interval representation. In two sections we analyze the general case and the up-growing one. The argument in the latter setting is much more complicated but it ends with a surprising result settling the value of the game in this variant to be $\lfloor \frac{1+\sqrt{5}}{2} w \rfloor$. Dealing with the lower bound (i.e., strategy for Spoiler) we significantly use the golden ratio rule. Then we show that all other Spoiler's forcing constructions can be reduced to a canonical form and indeed nothing better than $\lfloor \frac{1+\sqrt{5}}{2} w \rfloor$ can be achieved.

3.1. Non up-growing case

We start with a rather simple argument for the general case, when Spoiler presents points not necessarily in an up-growing order.

THEOREM 3.1. *The value of on-line chain partitioning game for semi-orders of width w presented without representation is $2w - 1$.*

PROOF. Having upper bound provided by greedy algorithm (see Fact 2.3) all we need to do is to present matching Spoiler's strategy. This strategy looks as follows:

(i) Present two antichains A and B, both consisting of w points in such a way that $A < B$, i.e., all points from A are below all points from B. If Algorithm uses $2w - 1$ or more colors, the construction is finished. Otherwise, suppose that k colors ($2 \leqslant k \leqslant w$) are used twice, once in A on $a_1, \ldots, a_k$ and once in B, on $b_1, \ldots, b_k$ respectively so that a_i and b_i have the same color.

(ii) Present $k - 1$ incomparable points $x_1, \ldots, x_{k-1}$ such that the only comparabilities are $a_1, \ldots, a_i < x_i < b_{i+1}, \ldots, b_k$. Their interval representation may look as in Figure 3.1. The width of obtained order is w. It is easy to verify that in such a setting Algorithm is forced to use $2w - 1$ chains as each x_i has to get a new color.

□

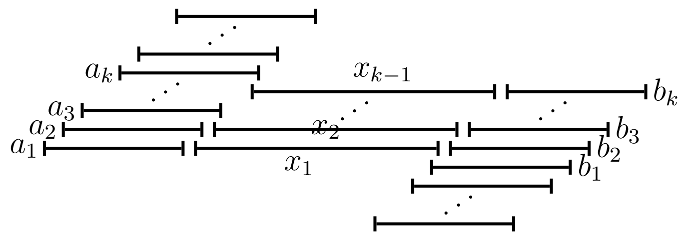

FIGURE 3.1. Strategy for Spoiler forcing Algorithm to use $2w - 1$ colors on semi-order of width w

3.2. Up-growing case

In the following we deal with on-line chain partitioning game for up-growing semi-orders. Let val-semi-upg(w) be the value of this game for orders of width at most w.

First, in Subsection 3.2.1 we introduce the concept of natural strategies for Algorithm, called natural algorithms. We show that they give the best possible values for the number of chains needed. Next two subsections present the lower bound of $\lfloor \frac{1+\sqrt{5}}{2} w \rfloor$ for val-semi-upg(w). In the analysis of Spoiler's strategy it would be of great help to know that Algorithm may respond with a natural coloring only. The crucial Subsection 3.2.4 is devoted to a reduction of any Spoiler's strategy witnessing val-semi-upg(w) to a normalized one. In a sequence of steps we modify a given strategy for Spoiler to bring it into a form which is very similar to the form of the lower bound strategy. For each step we prove that this modification does not reduce the number of enforced colors. Staying with the normalized form the desired result is obtained by analyzing the parameters of the final strategy (Subsection 3.2.5). Essentially this analysis amounts in solving a linear program whose optimal solution is the golden ratio

To emphasize the on-line nature of semi-order $\mathbf{P} = (P, \leqslant)$ presented by Spoiler and to make it handy for some modifications we will often write $P = (p_1, \ldots, p_n)$, where the sequence agrees with the order of presentation. Now, the up-growing condition for $\mathbf{P}$ means that $(p_1, \ldots, p_n)$ is a linear extension of $\mathbf{P}$.

3.2.1. Natural algorithms. For the better understanding of the idea of a natural algorithm for chain partitioning of an up-growing semi-order we start with the following example. Consider semi-order $\mathbf{P} = (P, \leqslant)$ with $P = (a, b, c, d, e)$ and the coloring $\Gamma : P \setminus \{e\} \to \mathbb{N}$ as shown in Figure 3.2. Point e may be colored with a new color (say, with color 4) or with one of the colors already used, i.e., with an old color. In the latter case Algorithm may choose between 2 and 3. (We say that color α is valid for a new point x extending an already colored poset $\mathbf{P}$ if x dominates all points in $\mathbf{P}$ colored with α). We claim that among the valid colors 2 and 3 defining $\Gamma(e) = 3$ is the better choice.

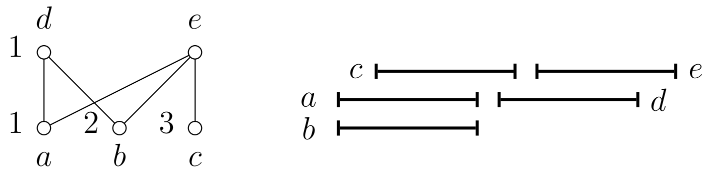

FIGURE 3.2. Poset **P** with its unit interval representation and coloring Γ of points a, b, c, d

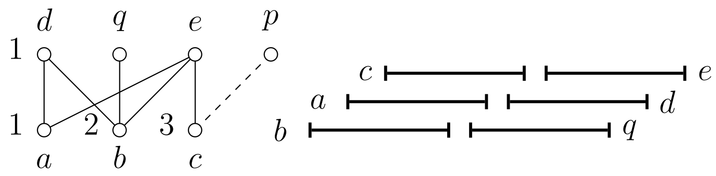

FIGURE 3.3. Point q may be presented by Spoiler in the future, point p can not.

Indeed, any future point p presented by Spoiler and dominating c also dominates b (otherwise, **P** would have a $(\mathbf{2}+\mathbf{2})$ configuration which is forbidden in semi-orders). On the other hand, Spoiler may play q greater than b but remaining incomparable to c (see Figure 3.3). Hence, using the color of c for e leaves more coloring options for the future. It turns out that whenever the colors of two points x and y are valid and $x \prec y$ (in the sense of (1) from page 24), then it is safer to use the color of y.

DEFINITION. Let p be a new point presented by Spoiler and let validTops(p) denote the set of points in $p{\downarrow}$ being the tops of colors valid for p. More formally, $\text{validTops}(p) = \{x < p : x = \text{top}(\gamma)$ for some color γ at the moment when p is presented$\}$. Point p is naturally colored if the following hold:

(i) if $\text{validTops}(p) \neq \emptyset$, then p gets the color of some q which is $\prec$–maximal in validTops(p),

(ii) if $\text{validTops}(p) = \emptyset$, then p receives any new color.

A coloring is natural if every point is colored naturally. A coloring algorithm is natural if it produces natural coloring.

Note that the natural color for a new point p need not be unique. However, all $\prec$-maximal points in validTops(p) have the same up- and down-sets and as such they are indistinguishable by data used by natural algorithm. Hence, a natural algorithm is free to choose any color from the set of equivalent candidates. Due to this observation we may abuse notation and frequently speak of 'the natural coloring' although there may be several of them.

The importance of natural colorings is given by the following theorem, in which by an optimal strategy we mean one that uses the smallest possible number of colors.

THEOREM 3.2. *Natural algorithms are optimal strategies for Algorithm in on-line chain partitioning game of up-growing semi-orders.*

Given an algorithm A, an on-line order $\mathbf{P} = (P, \leqslant)$ and $X \subseteq P$ we let $A(X)$ denote the set of colors used by A on X. The theorem follows from the next claim.

CLAIM 3.3. For any on-line algorithm A there is a natural on-line algorithm B such that $|B(P)| \leqslant |A(P)|$ for all up-growing semi-orders $\mathbf{P} = (P, \leqslant)$.

PROOF. To prove the Claim we construct inductively a sequence $A^{(k)}$ of algorithms such that $A^{(0)} = A$ and $A^{(k)}$ naturally colors the first k points of the up-growing semi-order $\mathbf{P}$. Moreover, $A^{(k+1)}$ is a modification of $A^{(k)}$ behaving exactly the same way on first k points as $A^{(k)}$ and the condition $|A^{(k)}(P)| \leqslant |A(P)|$ will be kept.

Suppose $A^{(k)}$ is defined. To define $A^{(k+1)}$, let p be the $(k+1)$-th point presented by Spoiler. There are three possibilities:

Case 1. $A^{(k)}$ colors p naturally.
Define $A^{(k+1)}(p) = A^{(k)}(p)$ and $A^{(k+1)}(u) = A^{(k)}(u)$ for any future point u presented by Spoiler.

Case 2. $A^{(k)}$ assigns to p a new color γ although some natural algorithm would have used an old color of the form $A^{(k)}(q)$ for some $q \in \text{validTops}(p)$.
Define $A^{(k+1)}(p) = A^{(k)}(q)$ and extend the coloring $A^{(k+1)}$ by interchanging colors γ and $A^{(k)}(q)$ for points arriving later, i.e., for any point u introduced by Spoiler after p put:

$$A^{(k+1)}(u) = \begin{cases} A^{(k)}(q), & \text{if } A^{(k)}(u) = \gamma, \\ \gamma, & \text{if } A^{(k)}(u) = A^{(k)}(q), \\ A^{(k)}(u), & \text{otherwise.} \end{cases}$$

Clearly, $\left|A^{(k+1)}(P)\right| \leqslant |A^{(k)}(P)|$. It remains to show that $A^{(k+1)}$ defines a proper coloring of $\mathbf{P}$, i.e., that the sets $\left\{u \in P : A^{(k+1)}(u) = \gamma\right\}$ and $\left\{u \in P : A^{(k+1)}(u) = A^{(k)}(q)\right\}$ are indeed chains in $\mathbf{P}$. The first set is obviously a chain as it is a subset of the chain colored with $A^{(k)}(q)$. The second one is a sum of two chains $\left\{u \leqslant q : A^{(k)}(u) = A^{(k)}(q)\right\}$ and $\left\{u \geqslant p : A^{(k)} = \gamma\right\}$ first of which lies completely below the other as $q < p$.

Case 3. $A^{(k)}$ assigns to p an old, non-natural color, i.e., in already presented poset there exist $r, q \in \text{validTops}(p)$ such that q is $\prec$-maximal in validTops(p), $r \prec q$ and $A^{(k)}(p) = A^{(k)}(r)$.
As in Case 2 define $A^{(k+1)}(p) = A^{(k)}(q)$ and exchange colors $A^{(k)}(r)$ and $A^{(k)}(q)$ for all points appearing after p (see Figure 3.4). Clearly,

p is naturally colored by $A^{(k+1)}$ and $|A^{(k+1)}(P)| = |A^{(k)}(P)|$. It remains to show that $A^{(k+1)}$ defines a proper coloring of $\mathbf{P}$. Again, the set $\{u \in P : A^{(k+1)}(u) = A^{(k)}(q)\}$ is a chain as this is a sum of chains $\{u \leqslant q : A^{(k)}(u) = A^{(k)}(q)\}$ and $\{u \geqslant p : A^{(k)}(u) = A^{(k)}(r)\}$. Since $\mathbf{P}$ is a semi-order and $r \prec q$ at the moment when p is presented we get that $r \prec q$ in $\mathbf{P}$ at any future stage of the game and hence $r < u$ whenever $q < u$ for all $u \in P$. This shows that the set $\{u : A^{(k+1)}(u) = A^{(k)}(r)\}$ being a sum of $\{u \leqslant r : A^{(k)}(u) = A^{(k)}(r)\}$ and $\{u > q : A^{(k)}(u) = A^{(k)}(q)\}$ is a chain.

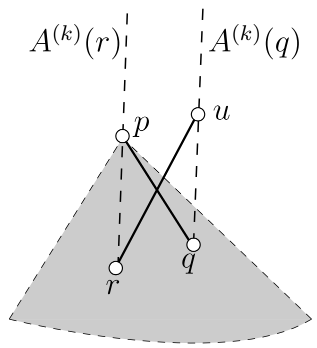

FIGURE 3.4. Exchanging colors $A^{(k)}(r)$ and $A^{(k)}(q)$

□

3.2.2. The lower bound: Fibonacci width. Let φ denote the golden ratio, i.e., $\varphi = \frac{1+\sqrt{5}}{2}$. In the following we develop a strategy for Spoiler that forces Algorithm to use $\lfloor \varphi \cdot w \rfloor$ chains for an up-growing semi-order of width w. We assume that Algorithm responds forming a natural coloring. The strategy for Spoiler is presented in two steps. First we assume that the width of the semi-order $\mathbf{P}$ is given by a Fibonacci number F_{2k+1}. In this case Spoiler will enforce F_{2k+2} chains. In a second step (Subsection 3.2.3) the strategy is analyzed for orders of arbitrary width.

i	0	1	2	3	4	5	6	7	8	9	...
F_i	0	1	1	2	3	5	8	13	21	34	...

TABLE 3.1. Fibonacci sequence

Spoiler builds an order $\mathbf{P} = (P, \leqslant)$ of width F_{2k+1}. The height of $\mathbf{P}$ will be at most 3, (for $F_1 = 1$ it is 1, for $F_3 = 2$ it is 2, for all others it is 3) therefore there is a canonical partitioning $P = A \cup B \cup C$ such that $A = \min(P)$ and $B = \min(P - A)$. The points of $\mathbf{P}$ are presented in packages such that the presentation sequence has the following structure

$$P = (A, B_0, B_1, C_1^0, C_1^1, B_2, C_2^0, C_2^1, C_1^2, B_3 \dots, \\ \dots, B_{k-1}, C_{k-1}^0, C_{k-1}^1, C_{k-2}^2, \dots, C_1^{k-1}, B_k).$$

Here, as suggested by our notation $B = B_0 \cup B_1 \cup \ldots \cup B_k$ and C is the sum of all the C_i^j's. We next describe $k+1$ phases of the construction of **P**.

Phase 0. Spoiler presents the antichain A of size F_{2k+1}. Algorithm uses F_{2k+1} colors.

Phase 1. Spoiler presents a set B_0 of points, such that $|B_0| = F_{2k-1}$ and $B_0 > A$ which means that $b > a$ for all $b \in B_0$ and $a \in A$. Algorithm responds in a natural way and uses colors which have already been used in A. Let $A_0 \subseteq A$ be the set of points whose colors have been used in B_0. Clearly, $|A_0| = F_{2k-1}$. Now Spoiler presents $B_1 > A_0$ with $|B_1| = F_{2k-1}$ (see Figure 3.5). Algorithm has to introduce F_{2k-1} new colors for B_1.

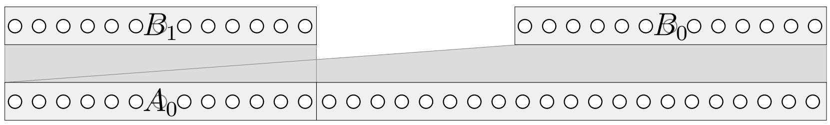

FIGURE 3.5. Order of width 34, Phase 1, $|B_0| = |B_1| = 13$

Phase 2. Spoiler presents a set C_1^0 of points such that $C_1^0 > A \cup B_1$ and $|C_1^0| = F_{2k-3}$. To color C_1^0 the natural Algorithm will use colors from a set $B_1^0 \subseteq B_1$ with $|B_1^0| = F_{2k-3}$. Now Spoiler presents $C_1^1 > A \cup B_1^0$ of size $|C_1^1| = F_{2k-3}$. Algorithm uses colors from a subset $A_1 \subseteq A - A_0$ with $|A_1| = F_{2k-3}$. Spoiler presents $B_2 > A_0 \cup A_1$ with $|B_2| = F_{2k-3}$ (see Figure 3.6). Algorithm has to introduce F_{2k-3} new colors for B_2.

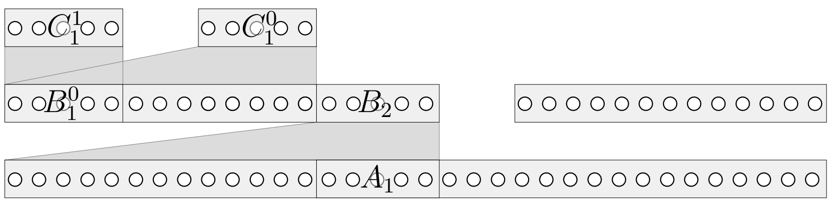

FIGURE 3.6. Order of width 34, Phase 2, $|C_1^0| = |C_1^1| = |B_2| = 5$

Now, let us informally describe the $(j+1)$-th Phase that generalizes Phase 2. The phase starts with the set C_j^0 dominating $A \cup B_1 \cup \ldots \cup B_j$, which forces natural Algorithm to pull some colors from B_j. This starts a cascade: The set $B_j^0 \subseteq B_j$ whose colors have been used for C_j^0 makes room for a set C_j^1 which fetches colors from points of $B_{j-1}^1 \subseteq B_{j-1}$.

This set B^1_{j-1} makes room for C^2_{j-1} which fetches colors from points of B^2_{j-2}. This continues until C^j_1 is presented. This last set is meant to fetch colors from A which makes room for a set B_{j+1}. However B_{j+1} has to receive new colors. All the sets introduced during this phase have size $F_{2k-2j-1}$. Figure 3.7 illustrates the 3rd phase on order of width 34. Now, we are ready for a formal and more detailed description of the $(j+1)$-th phase.

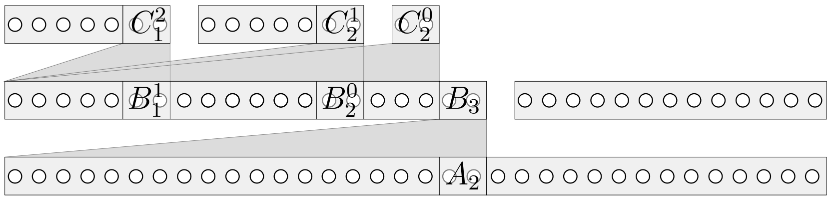

FIGURE 3.7. Order of width 34, Phase 3, $|C^0_2| = |C^1_2| = |C^2_1| = |B_3| = 2$

Phase $j+1$. All sets defined in this phase are declared to have size $F_{2k-2j-1}$. Spoiler presents $C^0_j > A \cup (B_1 \cup \ldots \cup B_j)$. To color C^0_j the natural Algorithm uses colors from some $B^0_j \subseteq B_j$. This starts a cascade and now, for $i = j, \ldots, 1$ Spoiler plays C^{j-i+1}_i such that $C^{j-i+1}_i > A \cup (B_1 \cup \ldots \cup B_{i-1}) \cup (B^0_i \cup B^1_i \cup \ldots \cup B^{j-i}_i)$. Algorithm, being natural, has to use colors of the points from some $B^{j-i+1}_i \subseteq B_i - (B^0_i \cup \ldots \cup B^{j-i}_i)$ to color C^{j-i+1}_i. For C^j_1 Algorithm uses colors from some $A_j \subseteq A - (A_0 \cup \ldots \cup A_{j-1})$. Finally, Spoiler presents $B_{j+1} > A_0 \cup \ldots \cup A_j$. This forces the Algorithm to introduce new colors since all the colors of predecessors of points in B_{j+1} have already been used for points in $B \cup C$.

The following identity for Fibonacci numbers is crucial for us:

$$F_1 + F_3 + F_5 + \ldots + F_{2r+1} = F_{2r+2}. \tag{2}$$

This implies that all together the sizes of B_j's do not exceed the width F_{2k+1}. Indeed,

$$\begin{aligned} |B| &= |B_0| + (|B_1| + |B_3| + \ldots + |B_k|) \\ &= F_{2k-1} + (F_{2k-1} + F_{2k-3} + \ldots + F_1) \\ &= F_{2k-1} + F_{2k} = F_{2k+1} = |A|. \end{aligned} \tag{3}$$

Similarly it follows that the sizes of all the sets that were put over some points in B_j but not in B_{j+1}, i.e., $C^0_j, C^1_j, \ldots, C^{k-j}_j$, sum up to the size of B_j:

$$\begin{aligned} |B_j\!\uparrow - B_{j+1}\!\uparrow| &= |C^0_j| + (|C^1_j| + \left|C^2_j\right| + \ldots + |C^{k-j}_j|) \\ &= F_{2k-2j-1} + (F_{2k-2j-1} + F_{2k-2j-3} + \ldots + F_1) \\ &= F_{2k-2j-1} + F_{2k-2j} = F_{2k-2j+1} = |B_j|\,. \end{aligned} \tag{4}$$

One can easily see that the downsets of the points from **P** are linearly ordered with respect to inclusion:

$$\begin{aligned}
&a{\downarrow} = \emptyset, \quad \text{for } a \in A,\\
&b{\downarrow} = A, \quad \text{for } b \in B_0,\\
&b{\downarrow} = A_0 \cup \ldots \cup A_{j-1}, \quad \text{for } b \in B_j,\ 1 \leqslant j \leqslant k,\\
&c{\downarrow} = A \cup (B_1 \cup \ldots \cup B_j), \quad \text{for } c \in C_j^0,\ 1 \leqslant j \leqslant k-1,\\
&c{\downarrow} = A \cup (B_1 \cup \ldots \cup B_{j-1}) \cup (B_j^0 \cup \ldots \cup B_j^{i-1}),\\
&\qquad\qquad \text{for } c \in C_j^i,\ 1 \leqslant j \leqslant k-1,\ 1 \leqslant i \leqslant k-j.
\end{aligned}$$

Therefore **P** is an interval order. To see that **P** is a semi-order suppose that **P** contains $(\mathbf{3}+\mathbf{1})$-configuration $d \parallel a, b, c$ with $a < b < c$. Since **P** has height at most 3 and $A < C$, the only option is that $d \in B$ while obviously $a \in A$, $b \in B$, $c \in C$. Let i, j be such that $b \in B_i$ and $d \in B_j$. Then it is easy to see that $a < d$ if $i \leqslant j$ and $d < c$ otherwise. This contradiction to $d \parallel a, c$ shows that **P** is $(\mathbf{3}+\mathbf{1})$-free so it is a semi-order.

In order to prove that the width of presented semi-order is $F_{2k+1} = |A|$ first observe that all antichains lie either in $A \cup B$ or in $B \cup C$ (since $A < C$). Consider an antichain $X \subseteq A \cup B$. If $X \subseteq A$ or $X \subseteq B$ there is nothing to prove (by (3)). Otherwise, let $j_0 \geqslant 1$ be the largest index such that $X \cap B_{j_0}$ is not empty. Now, with $X \subseteq (B_1 \cup \ldots \cup B_{j_0}) \cup A - (A_0 \cup \ldots \cup A_{j_0-1})$ and $|A_j| = |B_{j-1}|$ we have

$$|X| \leqslant (|B_1| + \ldots + |B_{j_0}|) + |A| - (|A_0| + \ldots + |A_{j_0-1}|) \leqslant |A|.$$

The proof that there is no antichain larger than B in $B \cup C$ is analogous. Consider an antichain $Y \subseteq B \cup C$. Again, for $Y \subseteq B$ there is nothing to prove. Let i, j be indexes of C_j^i with maximal possible downset such that $X \cap C_j^i \neq \emptyset$. Now, if $i = 0$ then

$$\begin{aligned}
Y \subseteq &(C_1^0 \cup C_1^1 \cup \ldots \cup C_1^{k-1}) \cup \ldots \cup (C_j^0 \cup \ldots \cup C_j^{k-j})\\
&\cup (B_{j+1} \cup \ldots \cup B_k \cup B_0)
\end{aligned}$$

and having $\sum_{i=0}^{k-j} \left|C_j^i\right| = |B_j|$ (by (4)) we obtain

$$|Y| \leqslant |B_1| + \ldots + |B_j| + |B_{j+1}| + \ldots + |B_k| + |B_0| = |B| = |A|.$$

For $i \geqslant 1$, in turn, we get

$$\begin{aligned}
Y \subseteq &(C_1^0 \cup C_1^1 \cup \ldots \cup C_1^{k-1}) \cup \ldots \cup (C_{j-1}^0 \cup \ldots \cup C_{j-1}^{k-j+1})\\
&\cup (C_j^1 \cup \ldots \cup C_j^i) \cup B_j - (B_j^0 \cup \ldots \cup B_j^{i-1})\\
&\cup (B_{j+1} \cup \ldots \cup B_k \cup B_0)
\end{aligned}$$

and again using $\sum_{i=0}^{k-j} \left|C_j^i\right| = |B_j|$ and $\left|B_j^l\right| = \left|C_j^{l+1}\right|$ we obtain $|Y| \leqslant |B| = |A|$.

The number of colors used by the natural algorithm is easily computed to be

$$|A| + |B_1| + \ldots + |B_k| = F_{2k+1} + F_{2k-1} + \ldots + F_1 = F_{2k+2}.$$

This shows the lower bound for Fibonacci width.

3.2.3. The lower bound: General width. We now refine the technique to prove the lower bound for orders of arbitrary width. The idea for the strategy remains the same as before, all we have to do is to adjust the sizes for the sets played in the consecutive phases.

As before after Phase j the sets:

$$A, B_0, B_1, C_1^0, C_1^1, B_2, \ldots, B_{j-1}, C_{j-1}^0, C_{j-1}^1, C_{j-2}^2, \ldots, C_1^{j-1}, B_j$$

are produced and $\left|C_{j-1}^0\right| = \left|C_{j-1}^1\right| = \left|C_{j-2}^2\right| = \ldots = \left|C_1^{j-1}\right| = |B_j|$. The only change will be with declaring their size. To do that we need the following values:

$$\begin{aligned} s_0 &= w, & r_0 &= \lfloor \varphi \cdot w \rfloor, \\ s_{j+1} &= \min(r_j, \lceil s_j(2-\varphi) \rceil), & r_{j+1} &= r_j - s_{j+1}. \end{aligned}$$

Now, we simply put:

$$\begin{aligned} |A| &= s_0, \\ |B_0| = |B_1| &= s_1, \\ |B_j| &= s_j, \qquad \text{for } j \geqslant 1. \end{aligned}$$

Note that the sequence s_j is eventually zero, i.e., there is k with $s_k > 0$ and $s_j = 0$ for $j > k$. Moreover,

$$s_{j+1} = \lceil s_j(2-\varphi) \rceil, \text{ for } 0 \leqslant j+1 \leqslant k-1, \tag{5}$$

i.e., the value r_j in the min-function comes into the play only in the size of the last cascade. Indeed, if the min-function chooses r_j for s_{j+1}, i.e., $s_{j+1} = r_j$ then $r_{j+1} = r_j - s_{j+1} = 0$ and $s_{j+2} = 0$.

With the following claim we show that these sizes allow the sets of a cascade to fit where they belong.

Claim 3.4. $s_j \geqslant s_{j+1} + r_j$, for $j = 0, \ldots, k-1$.

Proof. From $\varphi = 1 + 1/\varphi$ it follows that for every positive integer m we have:

$$\lfloor \varphi \cdot m \rfloor = \lfloor (1 + 1/\varphi)m \rfloor = m + \lfloor m/\varphi \rfloor, \tag{6}$$

$$m - \lfloor m/\varphi \rfloor = m + \lceil (1-\varphi)m \rceil = \lceil (2-\varphi)m \rceil. \tag{7}$$

Now, to see the Claim we induct on j.

$$\begin{aligned} s_1 + r_0 &= \lceil w(2-\varphi) \rceil + (\lfloor \varphi \cdot w \rfloor - w) && \text{by (6)} \\ &= \lceil w(2-\varphi) \rceil + \lfloor w/\varphi \rfloor && \text{by (7)} \\ &= w = s_0. \end{aligned}$$

The induction step is proved by following

$$
\begin{aligned}
s_{j+1} - r_{j+1} &= s_{j+1} - (r_j - s_{j+1}) \\
&= 2s_{j+1} - r_j \\
&\geqslant 2s_{j+1} - (s_j - s_{j+1}) && \text{(by ind.)} \\
&= (s_{j+1} - \lfloor s_{j+1}/\varphi \rfloor) + (\lfloor s_{j+1}/\varphi \rfloor + s_{j+1}) + s_{j+1} - s_j \\
&= \lceil s_{j+1}(2-\varphi) \rceil + \lfloor s_{j+1} \cdot \varphi \rfloor + s_{j+1} - s_j && \text{(by (7),(6))} \\
&\geqslant s_{j+2} + \lfloor s_{j+1}(1+\varphi) \rfloor - s_j \\
&= s_{j+2} + \lfloor \lceil s_j(2-\varphi) \rceil (1+\varphi) \rfloor - s_j && \text{(by (5))} \\
&\geqslant s_{j+2} + \lfloor s_j(2-\varphi)(1+\varphi) \rfloor - s_j \\
&= s_{j+2} + \lfloor s_j \rfloor - s_j \\
&= s_{j+2}.
\end{aligned}
$$

□

Expanding the statement of Claim 3.4 we get

$$
\begin{aligned}
s_j &\geqslant s_{j+1} + r_j \\
&= s_{j+1} + (s_{j+1} + r_{j+1}) \\
&\geqslant s_{j+1} + s_{j+1} + (s_{j+2} + r_{j+2}) \\
&\dots \\
&\geqslant s_{j+1} + \sum_{i=j+1}^{k} s_i.
\end{aligned}
$$

But this is nothing else than

$$
\begin{aligned}
|A| &\geqslant |B_0| + \sum_{i=1}^{k} |B_i|, \\
|B_j| &\geqslant |C_j^0| + \sum_{i=1}^{k-j} |C_j^i|, \text{ for } 1 \leqslant j \leqslant k-1.
\end{aligned}
$$

Hence, all cascades can indeed be played without violating the width. More precisely, the inequalities above can be used to prove that the width of presented construction is indeed w in an analogous way as (3), (4) in Fibonacci's width case.

After phase k we have $\min\{r_k, \lceil s_k\,(2-\varphi)\rceil\} = 0$ which is possible only if $r_k = 0$. By the definition of the s_j's and r_j's this implies $w + s_1 + \ldots + s_k = \lfloor \varphi \cdot w \rfloor$. Since the construction is tailored as to force s_j new colors in phase j we conclude that Spoiler has in total forced $\lfloor \varphi \cdot w \rfloor$ colors.

Theorem 3.5. *The value of on-line chain partitioning game for up-growing semi-orders of width w is at least $\lfloor \varphi \cdot w \rfloor$, i.e.,*

$$\text{val-semi-upg}(w) \geqslant \lfloor \varphi \cdot w \rfloor,$$

where $\varphi = \frac{1+\sqrt{5}}{2}$. □

3.2.4. The upper bound: Reductions. For a given w let $\mathbf{P} = (P, \leqslant)$ be an up-growing semi-order of width w which makes some natural algorithm use the maximal number of colors possible. Since natural algorithms are optimal val-semi-upg(w) colors are forced.

In the following we describe a new game by changing Spoiler's strategy while Algorithm will still behave naturally. This change in fact relies on a sequence of modifications that keep the following properties:

- $\mathbf{P}$ is an up-growing semi-order of width at most w,
- Γ is a natural coloring of $\mathbf{P}$ using val-semi-upg(w) colors.

After having modified $\mathbf{P}$ and Γ as to obey a certain set of properties (Properties (**a**)–(**m**)) we can describe some of the conditions as a system of inequalities whose solution will show that $\mathbf{P}$ could not force more than $\lfloor \varphi \cdot w \rfloor$ colors. This shows that the lower bound from Subsection 3.2.3 is also the upper bound for val-semi-upg(w).

The first modification is easy. Intuitively, when considering $\mathbf{P}$ we may ignore points that appear after already having val-semi-upg(w) colors used.

PROPERTY (**a**). *$P = (p_1, p_2, \ldots, p_n)$ and p_n is the first and only use of* val-semi-upg(w)*-th Γ-color.*

The next modification is slightly more substantial. We are going to prove that the height of $\mathbf{P}$ can be restricted to be at most 3.

PROPERTY (**b**). height($\mathbf{P}$) $\leqslant 3$.

Before showing that (**b**) can be achieved we need some preparation. In particular, we define the canonical partition of $\mathbf{P} = (P, \leqslant)$ into levels (antichains) $L_0, L_1, \ldots$ by:

$$L_0 = \min(P),$$
$$L_{i+1} = \min(P - (L_0 \cup \ldots \cup L_i)).$$

OBSERVATION 3.6. In a semi-order $\mathbf{P}$ we have:

(i) $x < y$ whenever $x \in L_i,\ y \in L_j$ and $i \leqslant j-2$,
(ii) $x \prec y$ whenever $x \in L_i,\ y \in L_j$ and $i < j$.

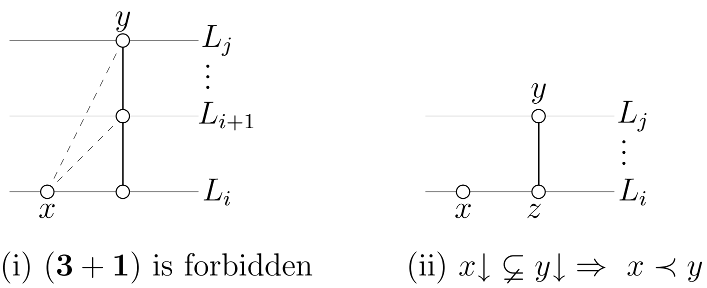

FIGURE 3.8. Proof of Observation 3.6

PROOF. To see (i) note that $x \not\prec y$ produces $(\mathbf{3}+\mathbf{1})$-configuration as presented by Figure 3.8. As to (ii), note that there is $z < y$ with $z \in L_i$. Now, $z \in y{\downarrow} - x{\downarrow}$, which implies in semi-order that $x{\downarrow} \subsetneq y{\downarrow}$ and therefore $x \prec y$. □

OBSERVATION 3.7. Let $\mathbf{P} = (P, \leqslant)$ be an up-growing semi-order. If Γ is a natural coloring of $\mathbf{P}$, then restricting Γ to $P - \min(P)$ yields a natural coloring of this smaller order.

PROOF. Pick $p \in P - \min(P)$. The proof is split into two cases:

Case 1. $\Gamma(p) = \gamma$ and, before p has been colored, $\mathrm{top}(\gamma) \in L_0$ in $\mathbf{P}$. The natural coloring Γ used for p a color γ with its top lying in L_0. Thus, there is no $\prec$-bigger available top for p. This, by Observation 3.6.(ii), gives $\mathrm{validTops}(p) \subseteq L_0$. This simply means that $\mathrm{validTops}_{P-L_0}(p) = \emptyset$ in the smaller poset, i.e., restricted to $P - L_0$. As $\Gamma_{|P-L_0}$ uses on p a new color it is natural on p.

Case 2. $\Gamma(p) = \gamma$ and, before p has been colored, $\mathrm{top}(\gamma) \in L_j$ in $\mathbf{P}$, for some $j \geqslant 1$.

Obviously, $\mathrm{top}(\gamma)$ remains $\prec$-maximal in $\mathrm{validTops}(p)$ in the smaller order. □

Now, we are ready to prove that Property (**b**) can be achieved.

PROOF OF PROPERTY (**b**). Observation 3.7 allows to decrease the number of levels as long as on the remaining levels all val-semi-upg(w) colors are used. We are going to show that highest three levels keep this number of colors.

Let h be the height of the last point p_n, i.e., $p_n \in L_h$. Since $\mathbf{P}$ is an up-growing order $p_n{\uparrow} = \emptyset$. This, together with the fact, that $L_i < L_j$ whenever $i \leqslant j - 2$ (by Observation 3.6.(i)), shows that L_j is empty for any $j \geqslant h + 2$. We define a new poset $\mathbf{Q} = (Q, \leqslant)$ which is a restriction of $\mathbf{P}$ to the three antichains $L_{h-1} \cup L_h \cup L_{h+1}$.

Iterating Observation 3.7 we get that the restriction of Γ on $\mathbf{Q}$ yields a natural coloring. All we have to prove is that $|\Gamma(Q)| = |\Gamma(P)|$. Consider the point p_n. If the coloring Γ of $\mathbf{P}$ would have a chain which is completely contained in $P \setminus Q$, then the color γ of this chain would have been valid for p_n as, by Observation 3.6.(i), $\mathrm{top}(\gamma) < p_n$. Since, according to (**a**), point p_n received a new Γ-color we conclude that the maximal element of every Γ-chain is contained in $\mathbf{Q}$.

It follows that the order $\mathbf{Q}$ forces as many colors as $\mathbf{P}$ does and we may replace $\mathbf{P}$ by $\mathbf{Q}$ which has height at most 3. □

By Property (**b**) we know that $\mathrm{height}(\mathbf{P}) \leqslant 3$. On the other hand from Example 1.7 we know that on the order of width w and height at most 2 Spoiler may force at most $\lfloor \frac{3}{2} w \rfloor$ colors. We have already seen in Theorem 3.5 that val-semi-upg$(w) \geqslant \lfloor \frac{1+\sqrt{5}}{2} w \rfloor$. Therefore for $w \geqslant 5$ the forcing poset $\mathbf{P}$ must have height 3. For $w \leqslant 4$ it is relatively easy to check that Spoiler may force at most $\lfloor \frac{3}{2}w \rfloor = \lfloor \frac{1+\sqrt{5}}{2}w \rfloor$ even playing on three levels. Thus, in the following we deal with $w \geqslant 5$ and $\mathrm{height}(\mathbf{P}) = 3$ with the levels L_0, L_1 and L_2. From now on, we denote these levels by A, B and C, respectively.

PROPERTY (**c**). $\Gamma(A) \nsubseteq \Gamma(B \cup C)$, *i.e., some colors are used only on level* A.

PROOF. Suppose to the contrary that $\Gamma(A) \subseteq \Gamma(B \cup C)$. We claim that the level A is redundant. Indeed, by Observation 3.7 we may exclude A from P and retain the natural coloring Γ on $B \cup C$. Clearly, the number of colors used does not change. □

COROLLARY 3.8.
(i) $A < C$,
(ii) New Γ-colors appear only on levels A and B, in other words, every Γ-chain has its minimal element in $A \cup B$.

PROOF. The fact that $A < C$ follows immediately from Observation 3.6.(i). To prove (ii) assume that there exists $c \in C$ which has obtained a new color by Γ. Since $A < c$ and Γ is natural, the point c could not use colors from $\Gamma(A)$. In particular we deduce that $\Gamma(A) \subseteq \Gamma(B \cup C)$ contradicting (**c**). □

PROPERTY (**d**). $|A| = w$.

PROOF. If $|A| < w$ then consider the set of new points $A' = \{q_1, \ldots, q_m\}$ such that $|A'| + |A| = w$. We construct an extended poset $\mathbf{Q} = (Q, \leqslant)$, with $Q = (q_1, \ldots, q_m, p_1, \ldots, p_n)$ where points in A' are made incomparable to A, dominated by $B \cup C$ and played by Spoiler at the very beginning. It may happen that coloring Γ on level B does, or does not, introduce at least m new colors. If it does then let $\gamma_1, \ldots, \gamma_m$ denote m first used such colors. Otherwise take for $\gamma_1, \ldots, \gamma_m$ all Γ-colors introduced on B and for the remaining γ's take completely new colors. Define Δ, a coloring of $\mathbf{Q}$:

$$\Delta(p) = \begin{cases} \Gamma(p), & \text{if } p \in P, \\ \gamma_i, & \text{if } p = q_i. \end{cases} \tag{8}$$

Since A' are $\prec$-minimal in $\mathbf{Q}$ at every moment of the game, further use of colors from $\Delta(A')$ happens only when there is no other option is natural. As Δ behaves exactly in this way, Δ is natural. □

PROPERTY (**e**). *Points from* A *are played at the beginning of the construction.*

PROOF. Let $P = (p_1, \ldots, p_i, a, p_{i+1}, \ldots, p_n)$ be the presentation sequence and $a \in A$. Consider presentation $(a, p_1, \ldots, p_i,\ p_{i+1}, \ldots, p_n)$. Since $\mathbf{P}$ is presented in an up-growing way a moment of thought reveals that the natural coloring Γ of the old sequence is also natural for the new sequence. Iterating this yields a sequence where A is played at the beginning. □

Corollary 3.9. For $P = (p_1, \dots, p_n)$ we have $p_n \in B$ and $p_n{\downarrow} \subsetneq A$.

Proof. By (**a**) and Corollary 3.8.(ii) $p_n \in A \cup B$. But for $p_n \in A$ Property (**e**) gives that $\mathbf{P}$ would be an antichain, contradicting height$(\mathbf{P}) = 3$. To see that $p_n{\downarrow} \subsetneq A$ suppose that $A < p_n$. This together with (**c**) gives that p_n could have obtained an old color from $\Gamma(A) - \Gamma(B \cup C)$ contradicting (**a**). □

The next three modifications deal with level B. First, we define

$$B_0 = \{p \in B : \ \Gamma(p) \in \Gamma(A)\}\,, \tag{9}$$

i.e., B_0 is the set of points of B which receive an old color.

Property (**f**). *$A < B_0$ and all points from B_0 are played right after A.*

Proof. Let $P = (A, p_1, \dots, p_i, b, p_{i+1}, \dots, p_n)$ be the presentation sequence of $\mathbf{P}$ and let b be the last element of this sequence which is in B_0.

Case 1. Suppose that $b{\uparrow} = \emptyset$. Let $\mathbf{Q} = (Q, \leqslant_{\mathbf{Q}})$ be a modification of $\mathbf{P}$ with the presentation sequence $Q = (A, b, p_1, \dots, p_i, p_{i+1}, \dots, p_n)$ and in which $A <_{\mathbf{Q}} b$.

Obviously, $\mathbf{Q}$ is an up-growing order. It is evident that the family of downsets $\{q{\downarrow}_{\mathbf{Q}} : q \in Q\}$ is still linearly ordered by inclusion, therefore $\mathbf{Q}$ is an interval order. Suppose that $\mathbf{Q}$ is not a semi-order, then, point b must contribute to a $(\mathbf{3}+\mathbf{1})$-configuration. Since b has gained additional comparabilities b cannot be the singleton in this configuration. On the other hand, being in Case 1, point b cannot be in the 3-chain either.

To note that Γ is also natural on $\mathbf{Q}$ observe that all points from A are $\prec$-maximal in validTops$_{\mathbf{Q}}(b)$. Thus, coloring b with any color from $\Gamma(A)$ is natural. The fact that $\Gamma(b)$ is reused earlier on B does not interfere with the naturality of Γ on the rest. Finally, point b in $\mathbf{P}$ could make some points in A-level $\prec_{\mathbf{P}}$-comparable, while they are not $\prec_{\mathbf{Q}}$-comparable. But such a change leaves all $\prec_{\mathbf{P}}$-maximal points in validTops$_{\mathbf{P}}(x)$ also $\prec_{\mathbf{Q}}$-maximal in validTops$_{\mathbf{Q}}(x)$, for all $x \in P$. This means that Γ always chooses color with $\prec_{\mathbf{Q}}$-maximal top in validTops$_{\mathbf{Q}}(x)$. Therefore Γ is natural.

Case 2. Now assume that $b{\uparrow} \neq \emptyset$. Since the above construction for $\mathbf{Q}$ may lead to a $(\mathbf{3}+\mathbf{1})$-configuration we need a more subtle argument. Note that $b{\downarrow} \subseteq p_n{\downarrow}$, as otherwise there would be a $(\mathbf{3}+\mathbf{1})$-configuration (see Figure 3.9).

This time $\mathbf{Q} = (Q, \leqslant_{\mathbf{Q}})$ is defined to be a modification of $\mathbf{P}$ with the presentation sequence $Q = (A, p_n, p_1, \dots, p_i, b, p_{i+1}, \dots, p_{n-1})$, where p_n

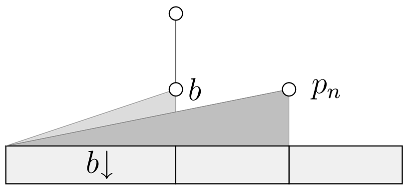

FIGURE 3.9. Property (**f**), $b\downarrow \subseteq p_n\downarrow$ in **P**

is immediately after A and such that $A <_{\mathbf{Q}} p_n$. The same argument as in Case 1 shows that the poset **Q** is an up-growing semi-order. However, the coloring Γ fails to be natural for **Q**, already at p_n. Thus, we define a new coloring Δ by interchanging colors of b and p_n and possibly of a successor of b from level C:

$$\Delta(p) = \begin{cases} \Gamma(b), & \text{if } p = p_n, \\ \Gamma(p_n), & \text{if } \Gamma(p) = \Gamma(b) \text{ and } p \in B \cup C, \\ \Gamma(p), & \text{otherwise.} \end{cases}$$

CLAIM 3.10. Δ is a natural coloring on **Q**.

PROOF. Observe that p_n is indeed naturally Δ-colored as it takes a color from some point in A while all points in A are $\prec_{\mathbf{Q}}$-maximal in $\text{validTops}_{\Delta,\mathbf{Q}}(p_n)$.

Now, we are going to show that assigning $\Delta(b)$ to be a new Δ-color is a natural behavior of Δ as it is justified by the emptiness of $\text{validTops}_{\Delta,\mathbf{Q}}(b)$.

Recall that we have chosen b as the last point of B_0 in the presentation sequence of **P**, i.e., the last point from B which received an old color by Γ. If b had only one available old color, i.e., $\left|\text{validTops}_{\Gamma,\mathbf{P}}(b)\right| = 1$, then we are done since this color has been taken by p_n in the Δ-coloring of **Q**. Now we show that the assumption $|\,\text{validTops}_{\Gamma,\mathbf{P}}(b)| > 1$ leads to a contradiction with the naturality of coloring Γ on **P**. Indeed, this assumption implies that after coloring b by Γ there was $a \in b\downarrow_{\mathbf{P}}$ whose Γ-color was not used in $B \cup C$. When Γ colors p_n this color was not available since p_n got a new Γ-color and $b\downarrow_{\mathbf{P}} \subseteq p_n\downarrow_{\mathbf{P}}$. Hence there is $c \in C$ with $\Gamma(c) = \Gamma(a)$. Now, Property (**c**) has to be witnessed by $d \in A$ such that $\Gamma(d)$ was not used for points in $B \cup C$. Note that $d \notin p_n\downarrow_{\mathbf{P}}$ as otherwise Γ would use $\Gamma(d)$ for p_n. On the other hand $a \in b\downarrow_{\mathbf{P}} \subseteq p_n\downarrow_{\mathbf{P}}$. Combining this we get $a \prec d$ witnessed by $b \in a\uparrow - d\uparrow$. Thus, $a \prec d$ holds at the moment when c arrives as c came after b. By Observation 3.8.(i) $d < c$. This shows that the natural coloring Γ would prefer $\Gamma(d)$ over $\Gamma(a)$ when coloring c. This contradiction proves that $|\,\text{validTops}_{\Gamma,\mathbf{P}}(b)| = 1$ and therefore b is naturally Δ-colored in **Q**.

The proof that all other points in **Q** are also naturally colored by Δ follows analogous argument in Case 1. This ends the proof of the claim. $\square$

As long as B_0 does not satisfy (**f**) we repeat the process described in Case 1 or Case 2 (whichever applies) to get the presentation sequence $(A, B_0, \ldots)$ with $A < B_0$. Note that the final **P** after these modifications may fail to give a new color to the latest point. But this can be repaired by going through modification (**a**) again. □

Since $p_n\!\downarrow\, \subsetneq A$ (by Corollary 3.9) and **P** is a semi-order, in particular $(\mathbf{3}+\mathbf{1})$-free, the fact that $A < B_0$ gives (see Figure 3.10)

$$B_0\!\uparrow\, = \emptyset. \tag{10}$$

This implies that after B_0 there is a non-empty group of points $B_1 \subseteq B$. Thus, we can split the sequence P into

$$P = (A, B_0, B_1, C_1, \ldots, B_k, C_k, B_{k+1}),$$

where the B_i's and the C_i's are non-empty groups of points from levels B and C, respectively. The last block is a B-block because the last point presented is in B (by (**a**)) The value $k \geqslant 1$ depends on **P** and its presentation. The following observation will help to enforce additional structure on B-blocks.

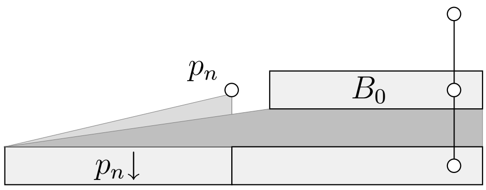

FIGURE 3.10. $B_0\!\uparrow\, = \emptyset$ because of forbidden $(\mathbf{3}+\mathbf{1})$

OBSERVATION 3.11 (Transposition of points). Let $P = (p_1, \ldots, p_n)$ be the presentation of an up-growing semi-order **P** of height at most 3 and let Γ be a corresponding natural coloring of **P**. If

(i) two consecutive points p_i and p_{i+1} got a new Γ-colors, or
(ii) $p_i \in C$, $p_{i+1} \in B$ and, before coloring p_i, $\mathrm{top}(p_i) \notin p_{i+1}\!\downarrow$,

then Γ remains a natural coloring for the modified up-growing presentation:

$$P' = (p_1, \ldots, p_{i-1}, p_{i+1}, p_i, p_{i+2} \ldots, p_n).$$

PROOF. Note that $p_{i+1} \not< p_i$ by the up-growing property. Now, if $p_i < p_{i+1}$ the assumption (i) gives that p_{i+1} could not get a new color in a natural coloring Γ. Also, under assumptions of (ii), the point p_i cannot be below p_{i+1} as $\mathrm{validTops}(p_i) \subseteq p_{i+1}\!\downarrow$. Therefore we get that p_i and p_{i+1} are incomparable and our switch remains up-growing. In the setting of (i) we have $\mathrm{validTops}(p_i) = \emptyset = \mathrm{validTops}(p_{i+1})$ and this is invariant under transposition. For (ii) we note that $\mathrm{validTops}(p_{i+1})$ is the same in P and P' and the top of the color chosen from $\mathrm{validTops}_P(p_i)$ is $\prec$-maximal in $\mathrm{validTops}_{P'}(p_i)$ as well. □

PROPERTY (**g**). *$b{\downarrow} \subsetneq b'{\downarrow}$ for $b \in B_i$, $b' \in B_j$ with $1 \leqslant i < j \leqslant k+1$.*

PROOF. Let $b \in B_i$ and $b' \in B_j$ do not satisfy the property. Since $\mathbf{P}$ is an interval order $b'{\downarrow} \subseteq b{\downarrow}$. We show how to move b' from B_j to B_i using a sequence of transpositions described in Observation 3.11. Iterating this procedure we reach a situation we are looking for.

Let p be the point that precedes b' in $\mathbf{P}$, i.e.,$P = (\dots, b, \dots, p, b', \dots)$. We show how to swap p and b'. Obviously $p \in B \cup C$. Assume first that $p \in B$. Since $p, b' \in B - B_0$ we invoke (**f**) to note that p and b' received a new Γ-color. Now, Observation 3.11.(i) allows us to swap p and b'.

Now assume that $p \in C$. Let q be the point from which p got its color. If $q \in B$ then q is incomparable with b' and we are allowed to swap p and b' by Observation 3.11.(ii). On the other hand for $q \in A$ we claim that $q \notin b'{\downarrow}$. As previously $b \in B - B_0$ gives that b got a new color, hence $q \notin b{\downarrow}$. This, by our assumption $b'{\downarrow} \subseteq b{\downarrow}$ gives $q \notin b'{\downarrow}$. Hence, we may swap p and b' again by Observation 3.11.(ii). □

PROPERTY (**h**). *$b{\downarrow} = b'{\downarrow}$ for $b, b' \in B_i$.*

PROOF. For $i = 0$ this is obvious, as $b{\downarrow} = A = b'{\downarrow}$ by Property (**f**). Fix $i > 0$ and suppose to the contrary that the downsets $b{\downarrow}$ with b ranging over B_i are not all equal. Let $b' \in B_i$ be such that $b{\downarrow} \subseteq b'{\downarrow}$ for all $b \in B_i$. Define an order $\mathbf{Q} = (P, \leqslant_{\mathbf{Q}})$ with $\leqslant_{\mathbf{Q}}$ being the same as $\leqslant_{\mathbf{P}}$ except for the points from B_i whose sets of predecessors in $\leqslant_{\mathbf{Q}}$ are enlarged to $b'{\downarrow}_{\mathbf{P}}$. We claim that we may replace $\mathbf{P}$ by $\mathbf{Q}$ while keeping the same coloring Γ. We prove here only non obvious conditions:

1) $\mathbf{Q}$ is $(\mathbf{2}+\mathbf{2})$-free.

This is equivalent to the fact that the downsets in $\mathbf{Q}$ are weakly ordered by inclusion, i.e., $q{\downarrow}_{\mathbf{Q}} \subseteq q'{\downarrow}_{\mathbf{Q}}$ or $q{\downarrow}_{\mathbf{Q}} \supseteq q'{\downarrow}_{\mathbf{Q}}$ for $q, q' \in P$. Our initial semi-order $\mathbf{P}$ has that property. Observe that the only points that have modified downsets when passing from $\mathbf{P}$ to $\mathbf{Q}$ are in B_i since all points above some of them already dominate A, as $A <_{\mathbf{P}} C$ (by Corollary 3.8.(i)). Moreover, all of modified downsets are equal $b'{\downarrow}_{\mathbf{P}}$, i.e., they are equal to some downset in $\mathbf{P}$. Thus, all we have done with this modification is that we narrowed the family of downsets and therefore they are still weakly ordered in $\mathbf{Q}$.

2) $\mathbf{Q}$ is $(\mathbf{3}+\mathbf{1})$-free.

Suppose to the contrary there exist $q_1, q_2, q_3, q_4 \in P$ such that $q_1 < q_2 < q_3$ and q_4 being incomparable with q_1, q_2, q_3 (see Figure 3.11). Since $\mathbf{P}$ is $(\mathbf{3}+\mathbf{1})$-free, the forbidden configuration must be formed by one of the extra edges in $\mathbf{Q}$ and obviously $q_2 \in B_i$. Recall that $A < C$ and therefore $q_4 \in B$. But $q_4 \notin B_{i+1} \cup \ldots \cup B_k \cup B_{k+1} \cup B_0$, as otherwise we would have $q_1 \in q_2{\downarrow}_{\mathbf{Q}} = b'{\downarrow}_{\mathbf{P}} \subseteq q_4{\downarrow}_{\mathbf{P}} = q_4{\downarrow}_{\mathbf{Q}}$, where the middle inclusion follows from (**g**). This would contradict $q_1 \parallel_{\mathbf{Q}} q_4$. Moreover, $q_4 \notin B_i$, as otherwise $q_1 \in q_2{\downarrow}_{\mathbf{Q}} = q_4{\downarrow}_{\mathbf{Q}}$, again contradicting $q_1 \parallel_{\mathbf{Q}} q_4$. Thus, we get $q_4 \in B_1 \cup \ldots \cup B_{i-1}$ and by (**g**) for $\mathbf{P}$ there must exist

$a \in A$ with $a \in q_2{\downarrow_{\mathbf{P}}} - q_4{\downarrow_{\mathbf{P}}}$. It is now easy to check that the quadruple $\{a < q_2 < q_3\} \parallel q_4$ forms a $(\mathbf{3}+\mathbf{1})$-configuration in $\mathbf{P}$ (see Figure 3.11), a contradiction.

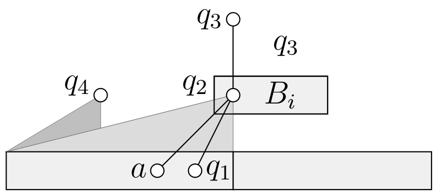

FIGURE 3.11. Property (**h**), $\mathbf{Q}$ is $(\mathbf{3}+\mathbf{1})$-free

3) Γ is natural on $\mathbf{Q}$.
When coloring $b \in B_i$ in $\mathbf{Q}$ the only old Γ-colors that might be assigned to b have to be in $\text{validTops}_{\mathbf{Q}}(b) = \text{validTops}_{\mathbf{P}}(b') = \emptyset$ (the latter equality follows from the fact that b' has a new Γ-color and Γ is natural in $\mathbf{P}$). Thus, also in $\mathbf{Q}$, new colors should be assigned to all points in B_i, exactly as Γ does. This shows that Γ behaves naturally when coloring B_i in $\mathbf{Q}$.

Our modification of the order relation could also change $\prec$-ordering. This could imply that some points which are naturally Γ-colored in $\mathbf{P}$ are not naturally Γ-colored in $\mathbf{Q}$. This however is not the case since $\prec_{\mathbf{Q}}$ is contained in $\prec_{\mathbf{P}}$ – the only change is that some points in $A \cup B_i$ which were $\prec_{\mathbf{P}}$-comparable are not $\prec_{\mathbf{Q}}$-comparable. This ensures that every $\prec_{\mathbf{P}}$-maximal point in a set $X \subseteq P$ remains $\prec_{\mathbf{Q}}$-maximal in X. Applying this observation to X ranging over the sets of the form validTops(x) we get that Γ remains natural on $\mathbf{Q}$. $\square$

Note that by now the sets A and B with its refinement $B_0, \ldots, B_{k+1}$ have the properties of the corresponding sets in the lower bound construction of Subsection 3.2.2. With few next modifications we establish corresponding properties on the level C of the poset.

Recall that poset $\mathbf{P}$ is presented in the order
$P = (A, B_0, B_1, C_1, \ldots, B_k, C_k, B_{k+1})$. For $p \in C$ define $j(p)$ as the largest number such that at least one point from the set $B_{j(p)}$ is covered by p, i.e.,

$$j(p) = \max\{i : B_i \cap p{\downarrow} \neq \emptyset\}.$$

Recall from (10) that $B_0{\uparrow} = \emptyset$ and therefore $j(p) \geqslant 1$ for all $p \in C$. Having $p > b$ for some $b \in B_{j(p)}$ we get $p > B_1 \cup \ldots \cup B_{j(p)-1}$. Otherwise, if $p \parallel b'$ for some $b' \in B_i$, $1 \leqslant i < j(p)$ then picking $a \in B_{j(p)}{\downarrow} - B_i{\downarrow}$ (which exists by Property (**g**)) we would have $\{a < b < p\} \parallel b'$ forming $(\mathbf{3}+\mathbf{1})$-configuration, as presented on Figure 3.12. Thus, we argued that

$$B_1 \cup \ldots \cup B_{j(p)-1} \subseteq p{\downarrow} \text{ for } j(p) \geqslant 2. \tag{11}$$

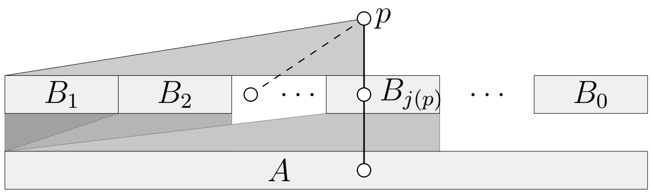

FIGURE 3.12. $B_1 \cup \ldots \cup B_{j(p)-1} \subseteq p\!\downarrow$ for $j(p) \geqslant 2$

DEFINITION.

(i) $U_j = \{p \in C : j(p) = j\}$.

(ii) A point $p \in C$ with $j = j(p)$ is a pulling point on B_j iff $\Gamma(p) \in \Gamma(B_j)$. The set of pulling points on B_j is denoted by W_j.

(iii) A point $p \in C$ with $j = j(p)$ which is not pulling is a cascading point on B_j. The set K_j of cascading points on B_j satisfies $K_j = U_j - W_j$.

(iv) Let F_j be the subset of B_j consisting of points whose colors are used in C, i.e.,

$$F_j = \{b \in B_j : \Gamma(b) \in \Gamma(C)\} .$$

(v) Let F be the subset of A consisting of points whose colors are used in $B \cup C$, i.e.,

$$F = \{a \in A : \Gamma(a) \in \Gamma(B \cup C)\} .$$

(vi) Denote the sizes of the introduced sets by corresponding lower case letters, i.e.,

$$b_j = |B_j|, \quad w_j = |W_j|, \quad k_j = |K_j|, \quad f_j = |F_j|, \quad f = |F|.$$

Our next property tells us that the relation of B_0 to A is now modeled by the relation of W_j to B_j.

PROPERTY (**i**). *$B_j < W_j$ and all points from W_j are played at the beginning of C_j, for $j = 1, \ldots, k$ (see Figure 3.13).*

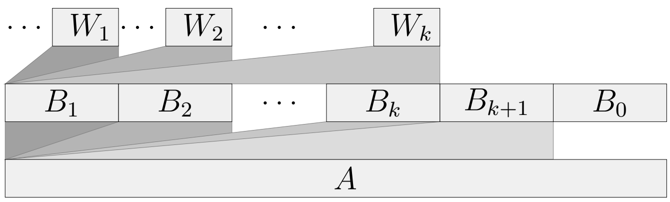

FIGURE 3.13. Place of W_j's (pulling points) inside **P**

Proof. Let $p \in W_j$. There is i with $p \in C_i$ and since p covers some point in B_j we know that $i \geqslant j$. We modify $\mathbf{P}$ to $\mathbf{Q}$ in order to have p presented at the beginning of the block C_j and to have $p > B_j$. By iterating this process, our Property (**i**) can be enforced.

To define $\mathbf{Q}$ we add comparabilities so that $B_j \subseteq p{\downarrow_{\mathbf{Q}}}$ and change the presentation sequence to

$$Q = (A, B_0, \ldots, C_{j-1}, B_j, p, C_j, \ldots, C_i \setminus \{p\}, B_{i+1}, \ldots, C_k, B_{k+1}).$$

Since $j = j(p)$ and by (11) we have $p{\downarrow_{\mathbf{Q}}} = A \cup B_1 \cup \ldots \cup B_j$. All of $A \cup B_1 \cup \ldots \cup B_j$ is played before p also in $\mathbf{Q}$ therefore the presentation of $\mathbf{Q}$ is up-growing.

Again to prove $(\mathbf{2}+\mathbf{2})$-free condition we argue that the family of downsets of $\mathbf{Q}$ is weakly ordered by $\subseteq$. Recall that the downsets in $\mathbf{P}$ are weakly ordered and we modified only the downset of p. This new downset fits well to the $\subseteq$-ordering. Indeed, whenever $q{\downarrow_{\mathbf{Q}}} \nsubseteq p{\downarrow_{\mathbf{Q}}} = A \cup B_1 \cup \ldots \cup B_j$ for $q \in P$ there must be $b \in B_{j+1} \cup \ldots \cup B_k$ such that $b \in q{\downarrow_{\mathbf{Q}}} = q{\downarrow_{\mathbf{P}}}$ (which means that q is on level C). Thus, we get $j(q) \geqslant j+1$ and by (11) also $B_1 \cup \ldots \cup B_j \subseteq q{\downarrow_{\mathbf{P}}}$. This together with the fact that $A < C$ (by Corollary 3.8) gives $p{\downarrow_{\mathbf{Q}}} \subseteq q{\downarrow_{\mathbf{Q}}} = q{\downarrow_{\mathbf{P}}}$. Thus, the family of downsets in $\mathbf{Q}$ are weakly ordered by inclusion and therefore $\mathbf{Q}$ is $(\mathbf{2}+\mathbf{2})$-free.

To finish the proof that $\mathbf{Q}$ is a semi-order suppose to the contrary that there is a $(\mathbf{3}+\mathbf{1})$-configuration in $\mathbf{Q}$. The semi-order $\mathbf{P}$ is $(\mathbf{3}+\mathbf{1})$-free therefore one of the new edges of p had to contribute to this forbidden configuration in $\mathbf{Q}$ so we have $\{q_1 <_{\mathbf{Q}} q_2 <_{\mathbf{Q}} p\} \parallel_{\mathbf{Q}} q_3$ and $q_2 \in B_j$. Since $j = j(p)$ in $\mathbf{P}$ there is $b \in B_j$ with $b <_{\mathbf{P}} p$. Now, $b{\downarrow_{\mathbf{P}}} = q_2{\downarrow_{\mathbf{P}}}$ (by (**h**)) implies $q_1 <_{\mathbf{P}} b$. We get a $(\mathbf{3}+\mathbf{1})$-configuration $\{q_1 <_{\mathbf{P}} b <_{\mathbf{P}} p\} \parallel_{\mathbf{P}} q_3$ in $\mathbf{P}$.

To see that Γ is natural on $\mathbf{Q}$ let us focus first on the point p. As p is presented in $\mathbf{Q}$ right after B_j and $p >_{\mathbf{Q}} B_j$ we have $B_j \subseteq \mathrm{validTops}_{\mathbf{Q}}(p)$. Moreover, all points in B_j at this moment are $\prec_{\mathbf{Q}}$-maximal in $\mathrm{validTops}_{\mathbf{Q}}(p)$ as they have the biggest downset among points in $p{\downarrow_{\mathbf{Q}}} = A \cup B_1 \cup \ldots \cup B_j$. Since Γ colors p with a color from B_j, this behavior is natural.

Our enlargement of $p{\downarrow_{\mathbf{P}}}$ to $p{\downarrow_{\mathbf{Q}}}$ might destroy some $\prec_{\mathbf{P}}$-comparabilities among the points in B_j. This time $\prec_{\mathbf{Q}}$ not necessarily is contained in $\prec_{\mathbf{P}}$ (like in the proof of (**h**)) as point p may spoil this rule. But observe that all sets of the form $\mathrm{validTops}(x)$ lies in $A \cup B$ and $\prec_{\mathbf{Q}|A\cup B}$ is indeed contained in $\prec_{\mathbf{P}|A\cup B}$. This secures the property that $\prec_{\mathbf{P}}$-maximal points in $\mathrm{validTops}(x)$ are also $\prec_{\mathbf{Q}}$-maximal. This immediately assures that Γ remains natural on $\mathbf{Q}$. □

Property (**j**). *A cascading point $p \in K_j$ dominates precisely those points of B_j whose color have already been used on level C when p is presented, i.e., for $j = 1, \ldots, k$, $p \in K_j$ and $b \in B_j$ we have $b < p$ iff* *there is $c \in C$ with $\Gamma(c) = \Gamma(b)$ and c precedes p in the presentation order.*

Proof. For a fixed j we are going to modify $\mathbf{P}$ to $\mathbf{Q}$ in order to have K_j as (**j**) states. To define $\mathbf{Q}$ we modify comparabilities so that each $p \in K_j$ dominates precisely these points in B_j whose Γ-colors has been already used on level C at the time when p is presented. First, observe that to achieve that we only have to add some comparabilities. Indeed, if p has dominated some $b \in B_j$ with $\Gamma(b) \notin \Gamma(C)$ at the moment when p is presented then $\Gamma(b)$ would be preferred over any color with top in $A \cup B_1 \cup \ldots \cup B_{j-1}$ as $A \cup B_1 \cup \ldots \cup B_{j-1} \prec_{\mathbf{P}} b$.

The proofs that $\mathbf{Q}$ is $(\mathbf{2}+\mathbf{2})$-free and $(\mathbf{3}+\mathbf{1})$-free follow the corresponding part in the proof of Property (**i**). There is a slight difference in the proof of the naturality of Γ on $\mathbf{Q}$. This time $\prec_{\mathbf{Q}}$ may not be contained in $\prec_{\mathbf{P}}$ even considering restriction to $A \cup B$. But this will happen only among points in B_j whose Γ-color is gone (on level C). Thus, such a change of $\prec$-ordering does not make a difference for a natural coloring. □

Corollary 3.12. For any two cascading points $k, k' \in K_j$

$$k{\downarrow} \subseteq k'{\downarrow} \text{ whenever } k \text{ precedes } k' \text{ in the presentation.}$$

Property (**k**). width$(B_j \cup U_j) = |B_j|$ *for* $j = 1, \ldots, k+1$.

Proof. Trivially width$(B_j \cup U_j) \geqslant |B_j|$. The equality holds for $j = k+1$ as $U_{k+1} = \emptyset$. Suppose that width$(B_j \cup U_j) > |B_j|$ for $j \leqslant k$. Since $B_j < W_j$ (by (**i**)) and $|B_j| \geqslant |W_j|$ (by the definition W_j takes colors from B_j) we have $U_j \neq W_j$ therefore $K_j \neq \emptyset$. Among points in K_j pick a point $p \in K_j$ with minimal downset $p{\downarrow_{\mathbf{P}}}$. Define $\mathbf{Q} = (Q, \leqslant_{\mathbf{Q}})$ such that p looses predecessors in B_j, i.e., $p{\downarrow_{\mathbf{Q}}} = p{\downarrow_{\mathbf{P}}} - B_j$.

Again the proof that $\mathbf{Q}$ remains a semi-order (i.e., $\mathbf{Q}$ is $(\mathbf{2}+\mathbf{2})$-free and $(\mathbf{3}+\mathbf{1})$-free) is much as in the proof of Property (**i**). To see that Γ is a proper coloring also on $\mathbf{Q}$ note that p obtains a Γ-color with a top from some point in $A \cup B_1 \cup \ldots \cup B_{j-1}$. This means that cutting $p{\downarrow_{\mathbf{P}}}$ to $p{\downarrow_{\mathbf{Q}}} = p{\downarrow_{\mathbf{P}}} - B_j$ we do not interfere Γ-coloring to be well defined. To prove that Γ is natural observe that $\prec_{\mathbf{P}|A \cup B}$ contains $\prec_{\mathbf{Q}|A \cup B}$, as the only difference point p can make is to $\prec_{\mathbf{P}}$-differentiate some points in B_j while they are $\prec_{\mathbf{Q}}$-incomparable. Thus, as before (in the proof of (**i**)) every $\prec_{\mathbf{P}}$-maximal point in validTops(x) remains $\prec_{\mathbf{Q}}$-maximal and therefore Γ remains natural on $\mathbf{Q}$.

However, deleting comparabilities we may have increased the width. The following claim shows that this is not the case.

Claim 3.13. width$(\mathbf{Q})$ = width$(\mathbf{P})$.

Proof. In order to get contradiction suppose that width$(\mathbf{Q}) >$ width$(\mathbf{P})$. Let X be an antichain in $\mathbf{Q}$ with $|X| =$ width$(\mathbf{Q})$. Clearly, X contains p and some $r \in B_j$ with $r <_{\mathbf{P}} p$. When passing from $\mathbf{P}$ to

$\mathbf{Q}$ point p has actually moved from U_j to U_{j-1}. For the sake of clarity we use the notion of the U_i's with respect to $\mathbf{P}$.

Since p dominates all points in A we deduce that $X \subseteq B \cup C$. Furthermore, since we put r to be in a downset of $p{\downarrow_{\mathbf{P}}}$, which was the minimal downset in U_j we get $r < U_j$. In fact, by (11) we know that $r < U_{j+1} \cup \ldots \cup U_k$ as well, and this holds in $\mathbf{Q}$ too. On the other hand since p dominated some points of B_j in $\mathbf{P}$ it has to dominate all of the $B_1 \cup \ldots \cup B_{j-1}$ and this holds also in $\mathbf{Q}$. Combining the facts above we deduce that (see Figure 3.14)

(12) $$X \subseteq B_0 \cup (B_j \cup \ldots \cup B_{k+1}) \cup (U_1 \cup \ldots \cup U_{j-1} \cup \{p\}).$$

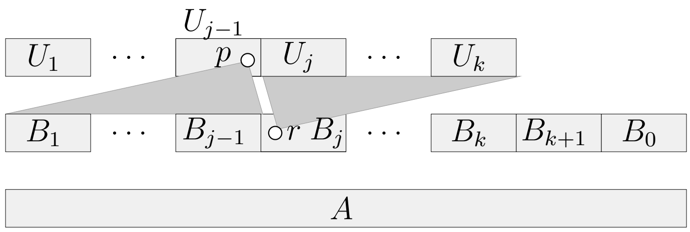

FIGURE 3.14. $r <_{\mathbf{Q}} (U_j \cup \ldots \cup U_k)$ and $p >_{\mathbf{Q}} (B_1 \cup \ldots \cup B_{j-1})$

Consider $X_0 = X \setminus \{p\}$. This is an antichain in $\mathbf{P}$. We will show that $|X_0| < \mathrm{width}(\mathbf{P})$.

Again, from the fact that $r < U_j$ we get $X_0 \cap (B_j \cup U_j) \subseteq B_j$ and with the assumption $\mathrm{width}(B_j \cup U_j) > |B_j|$ it follows that

(13) $$|X_0 \cap (B_j \cup U_j)| < \mathrm{width}(B_j \cup U_j) \text{ in } \mathbf{P}.$$

From (12) and comparabilities between the B_j's and the U_j's it follows that

(14) $$X_0 - (B_j \cup U_j) \text{ is pointwise incomparable to } B_j \cup U_j \text{ in } \mathbf{P}.$$

From (13) and (14) we obtain

$$\begin{aligned} |X_0| &= |X_0 \setminus (B_j \cup U_j)| + |X_0 \cap (B_j \cup U_j)| \\ &< |X_0 \setminus (B_j \cup U_j)| + \mathrm{width}(B_j \cup U_j) \\ &\leqslant \mathrm{width}(\mathbf{P}). \end{aligned}$$

Hence $\mathrm{width}(\mathbf{P}) \geqslant |X| = \mathrm{width}(\mathbf{Q})$ contradicting our assumption and finally proving $\mathrm{width}(\mathbf{P}) \geqslant \mathrm{width}(\mathbf{Q})$. On the other hand the antichain A witnesses that $\mathrm{width}(\mathbf{P}) = |A| \leqslant \mathrm{width}(\mathbf{Q})$ by Property (**d**). □

As we have already noticed, the point p was moved from U_j in $\mathbf{P}$ to U_{j-1} in $\mathbf{Q}$. If during this process p became a pulling point on B_{j-1} in $\mathbf{Q}$, Property (**i**) might be destroyed. In this case (**i**) needs to be reestablished as described in its proof. On the other hand if p became a cascading point on B_{j-1} in $\mathbf{Q}$, Property (**j**) might be destroyed and we fix it as in the proof of (**j**). In the case $j = 1$ point p in $\mathbf{Q}$ falls down to level B and since it has a Γ-color from A, Property (**f**) needs to be reestablished.

The size of U_j has decreased in $\mathbf{Q}$. Repeating this process we eventually establish (**k**). □

COROLLARY 3.14. $k_j \leqslant f_j$ for $j = 1, \ldots, k$.

PROOF. Since $K_j \cup (B_j - K_j\!\downarrow)$ is an antichain in $B_j \cup U_j$ we deduce with (**k**) that $|K_j| + |B_j - K_j\!\downarrow| \leqslant |B_j|$ and therefore

$$|K_j| \leqslant |B_j| - |B_j - K_j\!\downarrow| = |K_j\!\downarrow \cap B_j| \,.$$

On the other hand, by Property (**j**), cascading points over B_j are placed over point in B_j which Γ-color is already used on level C. This leads us to

$$K_j\!\downarrow \cap B_j \subseteq F_j .$$

Combining last two displays we get our Corollary. □

The next Property is meant to be a workhorse in calculating the upper bound. Loosely speaking it ensures that cascading points on B_j use colors from $\Gamma(B_{j-1})$ (for $j \geqslant 2$) and consequently the only points taking colors from A are those from $B_0 \cup K_1$. This fact will help us to bound the number of colors used by Γ on $\mathbf{P}$, i.e., to bound val-semi-upg(w).

PROPERTY (**l**). *$\Gamma(B_j) \not\subseteq \Gamma(C)$ for $j = 1, \ldots, k$, i.e., in each B_j there are colors not used on level C.*

PROOF. Let $\mathbf{P} = (P, \leqslant)$ and its coloring Γ satisfy (**a**)–(**k**) and fails to satisfy (**l**). Let s be the smallest index for which (**l**) does not hold:

$$s = \min \{ j \geqslant 1 : \ \Gamma(B_j) \subseteq \Gamma(C) \} \,.$$

We define $\mathbf{Q}$ and its natural coloring Δ such that the pair $(\mathbf{Q}, \Delta)$ satisfies (**a**)–(**k**), the size of the set B_s in $\mathbf{Q}$ is strictly smaller than the size of B_s in $\mathbf{P}$ and (**l**) holds for 'earlier' B-blocks, i.e., $\Gamma(B_j) \not\subseteq \Gamma(C)$ for $j = 1, \ldots, s-1$. We repeat this process until all bad B-blocks are removed and (**l**) is satisfied.

Let $b_s \in B_s$ be the point whose color was used on C as a last one among colors in $\Gamma(B_s)$ and let $p \in C$ be the point colored with $\Gamma(b_s)$. Depending on whether p is a pulling point or a cascading point, as well as on some other properties, we split the rest of the proof into cases.

Case 1. p is a pulling point on B_s.

Since all pulling points on B_s are played just after B_s (by (**i**)) and p is the last point in C that inherits its color from B_s we get $\Gamma(B_s) \subseteq \Gamma(W_s)$ and therefore $|W_s| = |B_s|$. Using (**k**) we get $K_s = \emptyset$.

We define new poset $\mathbf{Q}$ together with its presentation order by removing points b_s and p and by adding a new point q_s at the very end:

$$Q = (A, B_0, \dots, B_s \setminus \{b_s\}, C_s \setminus \{p\}, \dots, B_{k+1}, q_s),$$

$$q{\downarrow_{\mathbf{Q}}} = \begin{cases} B_{k+1}{\downarrow_{\mathbf{P}}}, & \text{if } q = q_s, \\ q{\downarrow_{\mathbf{P}}} \setminus \{b_s\}, & \text{otherwise.} \end{cases}$$

Note that, by (**h**), $q_s{\downarrow_{\mathbf{Q}}} = b{\downarrow_{\mathbf{P}}}$ for all $b \in B_{k+1}$. Now, color $\mathbf{Q}$ by:

$$\Delta(q) = \begin{cases} \Gamma(b_s), & \text{if } q = q_s, \\ \Gamma(q), & \text{otherwise.} \end{cases}$$

Clearly, Δ and Γ use the same number of colors as $\Delta(q_s) = \Gamma(b_s) = \Gamma(p)$.

To see that $\mathbf{Q}$ is $(\mathbf{2}+\mathbf{2})$-free note that downsets in $\mathbf{Q}$ are, up to the point b_s, the same as in $\mathbf{P}$. Therefore the inclusion weakly orders them as well. Now, we are going to argue that the poset $\mathbf{Q}$ remains $(\mathbf{3}+\mathbf{1})$-free. The only point that could contribute to a new $(\mathbf{3}+\mathbf{1})$-configuration is q_s. But since $q_s{\uparrow_{\mathbf{Q}}} = \emptyset$ we know that it can not be in a 3-chain. On the other hand there is no 3-chain completely incomparable to q_s in $\mathbf{Q}$. Indeed, such a chain had to start on one of the point in $A - B_{k+1}{\downarrow}$. On the other hand, since $B_1{\downarrow} \subseteq B_2{\downarrow} \subseteq \dots \subseteq B_{k+1}{\downarrow}$ (by (**g**)), the points from level B that lie over some point in $A - B_{k+1}{\downarrow}$ have to be in B_0. However, (10) says that $B_0{\uparrow} = \emptyset$, and therefore there is no room for a 3-chain.

To prove that $\text{width}(\mathbf{Q}) = \text{width}(\mathbf{P})$ we argue first that $\text{width}_{\mathbf{P}}(A \cup B) = \text{width}_{\mathbf{Q}}(A \cup B)$. Indeed, by (**d**), $\text{width}_{\mathbf{P}}(A \cup B)$ is witnessed by A and we deleted b_s with smaller downset than the added point q_s. Now, to show that $\text{width}(\mathbf{Q}) = \text{width}(\mathbf{P})$, it is enough to prove that Property (**k**) holds for $\mathbf{Q}$, i.e., $\text{width}(B_j \cup U_j) = |B_j|$ in $\mathbf{Q}$ for all j's. But passing from $\mathbf{P}$ to $\mathbf{Q}$ nothing has changed in $B_j \cup U_j$ for $j \neq s, k+1$. Looking at $B_s \cup U_s$, in turn, one point was deleted from B_s and one from U_s. Since $B_s < W_s$ and being in Case 1, we know that $W_s = U_s$ so that the Property (**k**) holds also for $j = s$. Moreover $U_{k+1} = \emptyset$ gives Property (**k**) for $j = k+1$.

Finally, the fact that Δ is natural on $\mathbf{Q}$ immediately follows from the naturality of Γ on $\mathbf{P}$.

Case 2. p is a cascading point on B_t for some $t > s$ and there are no cascading points on B_s.

Recall that $\Gamma(p) \in \Gamma(B_s)$ means that not all colors from $\Gamma(B_s)$ are used by W_s. Combining this with $K_s = \emptyset$ we have

$$|B_s| > |W_s| = |U_s|. \tag{15}$$

Define $\mathbf{Q}$ and Δ as in Case 1, i.e., points b_s and p are replaced by point q_s in B_{k+1}. All arguments from Case 1, but the one for Property (**k**), survive. For $j \neq s, t, k+1$ nothing has changed in $B_j \cup U_j$ and

Property (**k**) is obvious. Since we removed $p \in U_t$ while B_t was left unchanged, (**k**) holds there. The fact that (**k**) holds for $j = s$ follows directly from (15) and the fact that, by Property (**i**), $B_s < W_s$. Finally, for $j = k + 1$ recall that $U_{k+1} = \emptyset$.

Case 3. p is a cascading point on B_t for some $t > s$ and there are some cascading points on B_s, i.e., $p \in K_t$ and $K_s \neq \emptyset$.

The previous constructions may not apply as it can happen that $|B_s| = |U_s|$ in **P** and reducing the size of B_s without reducing corresponding U_s, like into Case 1, spoils (**k**) in $B_s \cup U_s$. Let k_s be the last (with respect to the presentation order) cascading point from K_s and let b_{s-1} be a point from which k_s got its Γ-color. By the choice of s we know that if $s > 1$ then there are some Γ-colors available for k_s in $\Gamma(B_{s-1})$, while for $s = 1$ there is some Γ-color available for k_s in A (by (**c**)). Obviously, one of these available colors has to be chosen by natural coloring Γ. Therefore, $b_{s-1} \in B_{s-1}$ for $s > 1$ and $b_{s-1} \in A$ for $s = 1$.

Case 3.1. k_s is played after p in **P**.

In this setting $\Gamma(b_s)$ is already used on level C at the time when k_s is presented. Define **Q** in which points b_s and k_s are removed and a new point q_s is added.

$$Q = (A, B_0, \ldots, B_s \setminus \{b_s\}, C_s \setminus \{k_s\}, \ldots, B_{k+1}, q_s),$$

$$q{\downarrow_{\mathbf{Q}}} = \begin{cases} B_{k+1}{\downarrow_{\mathbf{P}}}, & \text{if } q = q_s, \\ q{\downarrow_{\mathbf{P}}} \setminus \{b_s\}, & \text{otherwise.} \end{cases}$$

Define also a new coloring function on **Q**:

$$\Delta(q) = \begin{cases} \Gamma(b_s), & \text{if } q = q_s. \\ \Gamma(b_{s-1}), & \text{if } q = p, \\ \Gamma(q), & \text{otherwise.} \end{cases}$$

As in Case 1 we may argue that Δ and Γ use the same number of colors and **Q** remains a semi-order.

We are going to show that Δ is natural on **Q**. First of all we need an argument that $\Delta(p) = \Delta(b_{s-1}) = \Gamma(b_{s-1})$ is a natural behavior. Observe that at the moment when p arrives in **Q**:

- No point from $B_{s+1} \cup \ldots \cup B_t$ lies in $\text{validTops}_{\Delta,\mathbf{Q}}(p)$, as there were no such in $\text{validTops}_{\Gamma,\mathbf{P}}(p)$.
- All colors from $\Delta(B_s)$ have been already used on C, as b_s was the last point with Γ-top in B_s and b_s was removed when constructing **Q**. Thus $B_s \cap \text{validTops}_{\Delta,\mathbf{Q}}(p)$ is also empty.
- If $s > 1$ all points from B_{s-1} whose Δ-colors have not been used on C have the same downset and upset. Let $b, b' \in B_{s-1}$ be the points with Γ-tops at the considered moment of the game in **P**. The downsets of b and b' are equal

by Property (**h**). By (**i**) and (11) we get that W_{s-1} and all already presented points from $U_s, \ldots, U_k$ are in both upsets $b\uparrow, b'\uparrow$ at the moment. Thus, the only points which could differentiate the upsets $b\uparrow$ and $b'\uparrow$ are in K_{s-1}. But by (**j**) and the fact that b and b' are Γ-tops of their colors, we get that no point from K_{s-1} is in their upsets at the moment. As till this moment (arrival of the point p) Δ-tops in B_{s-1} in **Q** are exactly where respective Γ-tops in B_{s-1} in **P**, we are done.

- All points in A whose Δ-colors have not been used in $B \cup C$ have the same upsets (and obviously downsets).
  By (**f**) and Corollary 3.8.(i) we get that $A < B_0 \cup C$. Let $a \in A$ be a point with Γ-color not appearing elsewhere at the considered moment. As each point in $B_1 \cup \ldots \cup B_{k+1}$ gets a new color the already presented points in $B_1 \cup \ldots \cup B_{k+1}$ must be incomparable with a.

All together this imply that b_{s-1}, the point being Δ-top in **Q** at the time when p is presented, is $\prec_{\mathbf{Q}}$-maximal in $\text{validTops}_{\Delta,\mathbf{Q}}(p)$ and therefore Δ is natural when choosing $\Delta(b_{s-1})$ on point p in **Q**.

Note that any cascading point $k \in K_{s-1}$ that dominates b_{s-1} in **P** is played after k_s (otherwise, a natural coloring Γ could assign a better color $\Gamma(b_{s-1})$ to k and k would be a pulling point on B_{s-1}). Being in Case 3.1 we know that p is presented before k_s. Thus Δ-color of b_{s-1} is reused on level C (on point p) before arrival of any cascading point from K_{s-1} dominating b_{s-1}.

The facts above, together with the naturality of Γ, give that Δ colors naturally all points in **Q**.

To prove that the width of **Q** does not exceed the width of **P** we argue that (**k**) holds for $B_s \cup U_s$ in **Q**, i.e., $\text{width}(B_s \cup U_s) = |B_s|$ in **Q**. In fact we show that each antichain witnessing the width of $B_s \cup U_s$ in **P** includes b_s or k_s. First, we argue that each point from U_s dominating b_s must dominate the whole B_s. This is obvious for points from W_s as they dominate B_s by (**i**). By (**j**), in turn, we have that incoming cascading points on B_s dominate all points in B_s which Γ-colors are already used on C. But, by the definition of b_s, $\Gamma(b_s)$ is the last color with Γ-top in B_s and therefore all points from K_s dominating b_s must dominate the whole B_s. Therefore indeed each point in U_s dominating b_s dominates entire B_s. Now, consider a maximal antichain X in $B_s \cup U_s$ in **P** with $b_s \notin X$. This means that X has a point x dominating b_s but as we noted we must then have $B_s < x$. This means that $X \subseteq U_s$ and by the maximality of X we get $X = U_s$. Thus, deleting points b_s and any point from U_s (we deleted k_s) we decrease the width of $B_s \cup U_s$ as well as the size of B_s.

Case 3.2. p is played after k_s in **P**.

As in Case 3.1, we are heading for the poset $\mathbf{Q}$ in which points b_s and k_s could be replaced with q_s. This time, however, a more subtle construction is needed in order to retain the coloring from $\mathbf{P}$. Indeed, $\mathbf{P}$ may have a point $k \in K_{s-1}$ dominating b_{s-1} which is played before p. If k would be left in $\mathbf{Q}$ without further modifications, a natural color assigned to k could come from b_{s-1} instead of coming from earlier B-blocks.

Define the sequence of points $b_s, k_s, b_{s-1}, k_{s-1}, \ldots, k_r, b_{r-1}$ $(r \geqslant 1)$ as follows. Points b_s and k_s are already defined. Let b_i be the point, actually Γ-top, from which k_{i+1} got its Γ-color. Let $k_i \in K_i$ be the first point in presentation order dominating b_i. If such point k_i does not exist our sequence ends on b_i and $r = i + 1$, see Figure 3.15. From the minimality of s when defining B_s to be the first B-block violating (**l**) we have $b_i \in B_i$ for $i \geqslant r - 1$ and $i \neq 0$. Moreover, if $r = 1$ so that $k_1 \in K_1$ and b_0 are defined, k_1 takes Γ-color from A, as by the definition of K_1 there is no valid top for k_1 in B_1 while by (**c**) there must be some in A. Thus in this setting, $b_0 \in A$, see Figure 3.16.

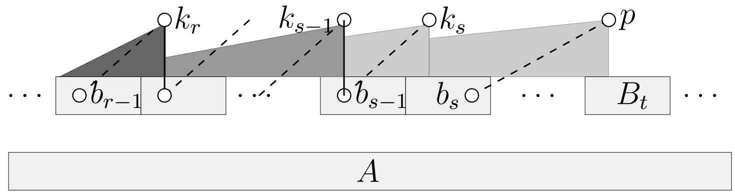

FIGURE 3.15. Property (**l**), Case 3.2.1, $r > 1$

Case 3.2.1. $r > 1$. (see Figure 3.15)

Observe that the cascade of points $k_s, \ldots, k_r$ does not pull any color from A to C. Intuitively, such a sequence of Spoiler's moves is redundant and may be omitted. We define $\mathbf{Q}$ in which the cascade $k_s, \ldots, k_r$ and the points b_s, p are not presented. In exchange for b_s a new point q_s is played. All remaining cascading points from K_j $(j = s-1, \ldots, r-1)$ are made incomparable with b_j. More formally, $Q = P - \{k_s, \ldots, k_r, b_s, p\} \cup \{q_s\}$ and q_s is played at the very end together with points from B_{k+1} and

$$q{\downarrow_{\mathbf{Q}}} = \begin{cases} B_{k+1}{\downarrow_{\mathbf{P}}}, & \text{if } q = q_s, \\ q{\downarrow_{\mathbf{P}}} - \{b_j\}, & \text{if } q \in K_j - \{k_j\}, j = s-1, \ldots, r, \\ q{\downarrow_{\mathbf{P}}} - \{b_s\}, & \text{otherwise.} \end{cases}$$

For our further arguments we need to take care that passing to $\mathbf{Q}$ and restricting the downsets of points from K_j to the form $q{\downarrow_{\mathbf{P}}} - \{b_j\}$, does not result in moving some of the q's to U_{j-1}. Suppose that there was $k \in K_j - \{k_j\}$ with $j = r, \ldots, s-1$ such that $k{\downarrow_{\mathbf{P}}} \cap B_j = \{b_j\}$. As k_j, by its definition, is the first (in the presentation order) point from K_j dominating b_j we get that k must be presented after k_j. Now Corollary 3.12 gives $k_j{\downarrow_{\mathbf{P}}} \cap B_j = \{b_j\}$ as well. This however, contradicts the local width (Property (**k**)) of $B_j \cup U_j$ in **P**. In conclusion we know that $k{\downarrow_{\mathbf{Q}}} \cap B_j \neq \emptyset$, i.e., every $k \in K_j - \{k_j\}$ in **P** remains a cascading point on B_j in **Q** as well.

We define a coloring Δ of **Q** by

$$\Delta(q) = \begin{cases} \Gamma(b_s), & \text{if } q = q_s, \\ \Gamma(q), & \text{otherwise.} \end{cases}$$

Note that Δ uses on **Q** the same number of colors as Γ on **P** – the Γ-colors of the deleted points $k_r, \ldots k_s$ are also used on the corresponding b_j's and the common Γ-color of b_s and p is used by Δ on q_s.

Again we argue that Δ is natural. The most significant difference of Δ comparing to Γ is that Δ-colors of $b_{r-1}, \ldots, b_{s-1}$ stay on level B and remain available for future points above. But this is exactly why we excluded b_j's from the downsets of K_j's in **Q**. With this note the naturality of Δ on **Q** is clear from the naturality of Γ on **P**.

As we introduced some new incomparabilities between points in B_j and U_j (for $j = r, \ldots, s-1$) we need a new proof that Property (**k**),i.e., $\text{width}(B_j \cup U_j) = |B_j|$, still holds. The proof for $j = s$ is exactly the same as in Case 3.1. We show that (**k**) holds for $B_r, \ldots, B_{s-1}$ in **Q**. Fix $j \in \{r, \ldots, s-1\}$. Since $|B_j| = \text{width}(B_j \cup U_j)$ in **P**, there exists an injective function $f : U_j \to B_j$ such that $u > f(u)$ for any $u \in U_j$ in **P**. If there is no $u \in U_j$ with $f(u) = b_j$ or such u exists and $u >_{\mathbf{Q}} b_j$ also in **Q** then f witnesses (**k**) for $B_j \cup U_j$ in **Q**. Otherwise, the point $k = f^{-1}(b_j)$ and k is cascading on B_j. The choice of k_j and Corollary 3.12 imply $k_j{\downarrow_{\mathbf{P}}} \subseteq k{\downarrow_{\mathbf{P}}}$. Therefore k can be matched with $f(k_j)$ in **Q** and this new injection witnesses $|B_j| = \text{width}(B_j \cup U_j)$ in **Q**.

Case 3.2.2. $r = 1$ (see Figure 3.16).

Unlike in Case 3.2.1, the cascade $k_s, \ldots, k_1$ pulls one color from A to C and therefore is not redundant in **P**. The idea is to replace points b_s, k_s with q_s and to shift the cascade after point p. This can be interpreted as replacing the cascade $k_s, k_{s-1}, \ldots, k_1$ with $p, k_{s-1}, \ldots, k_1$.

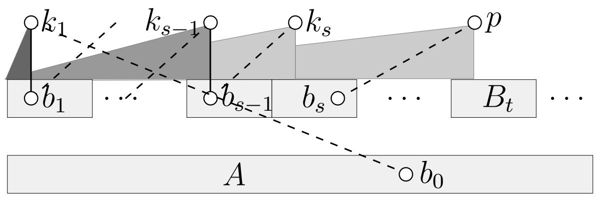

FIGURE 3.16. Property (**l**), Case 3.2.2, $r = 1$

We define **Q** in which:

- Points b_s and k_s are replaced with q_s, played in phase B_{k+1}. (Introducing q_s is intended to guarantee that the number of colors used by Γ on $\mathbf{P}$ is the same as the one used by Δ on $\mathbf{Q}$.)
- Cascade $k_{s-1}, \ldots, k_1$ is played at the end of the phase C_k (i.e., the last phase on the level C, in particular $k_{s-1}, \ldots, k_1$ are played after p and p is supposed to get the color from b_{s-1}); $k_j{\downarrow_{\mathbf{Q}}} = K_j{\downarrow_{\mathbf{P}}}$, $j = 1, \ldots, s-1$ (downsets of k_j's are extended in $\mathbf{Q}$ so that (**j**) is fulfilled).
- The remaining cascading points in $K_j - \{k_j\}$ ($j = 1, \ldots, s-1$) are made incomparable with b_j, i.e., $k{\downarrow_{\mathbf{Q}}} = k{\downarrow_{\mathbf{P}}} - \{b_j\}$. (This reserves the colors of b_{s-1} and $b_{s-2}, \ldots, b_1$ for p and the part $k_{s-1}, \ldots, k_2$ of the final cascade.)
- Points from the set B_j ($j = 1, \ldots, k$) are made incomparable with $b_0 \in A$, i.e., $b{\downarrow_{\mathbf{Q}}} = b{\downarrow_{\mathbf{P}}} - \{b_0\}$, for $b \in B_1 \cup \ldots \cup B_k$. (This reserves the color of b_0 for the point k_1 of the final cascade.)

The coloring Δ of $\mathbf{Q}$ is defined by $\Delta(q_s) = \Gamma(b_s)$, $\Delta(p) = \Gamma(b_{s-1})$ and $\Delta(q) = \Gamma(q)$ for all other q's. The proof of naturality follows the ones in the previous cases.

To prove that $\text{width}(B_j \cup U_j) = |B_j|$ for $j = 1, \ldots, s-1$ we again use a matching argument. Since $\text{width}(B_j \cup U_j) = |B_j|$ in $\mathbf{P}$ there is an injection $f : U_j \to B_j$ with $u >_{\mathbf{P}} f(u)$. Again, if there is no $k \in U_j$ such that $f(k) = b_j$ or such k exists and $k >_{\mathbf{Q}} b_j$ then f witnesses $\text{width}(B_j \cup U_j) = |B_j|$ in $\mathbf{Q}$. Otherwise we have $f(k) = b_j'$, $k >_{\mathbf{P}} b_j$ and $k \not>_{\mathbf{Q}} b_j$ which means that $k \in K_j - \{k_j\}$. Since k_j was the first (in the presentation order) point from K_j dominating b_j by Corollary 3.12 we get $k_j{\downarrow_{\mathbf{P}}} \subseteq k{\downarrow_{\mathbf{P}}}$. Thus in $\mathbf{Q}$ we can match k with $f(k_j)$ and k_j with b_j in $\mathbf{Q}$.

From Property (**k**) it follows that $\text{width}(B \cup C) = |B|$ in $\mathbf{Q}$. Thus, to prove that $\text{width}(\mathbf{Q}) = \text{width}(\mathbf{P})$ it is enough to show that $\text{width}(A \cup B) = |A|$ in $\mathbf{Q}$. Since $\text{width}(A \cup B) = |A|$ in $\mathbf{P}$ (by (**d**)) there is an injection $g : B \to A$ such that $b > g(b)$ in $\mathbf{P}$ for any $b \in B$. If there is no $b \in B$ such that $g(b) = b_0$ or such b exists and $b >_{\mathbf{Q}} b_0$, we match q_s with $g(b_s)$ and this new matching certifies the width. Otherwise, we have $b >_{\mathbf{P}} g(b) = b_0$ and $b \not>_{\mathbf{Q}} b_0$. Note that $B_1, \ldots, B_s$ are presented before k_s in $\mathbf{P}$, k_s before k_1 in $\mathbf{P}$ and k_1 got its Γ-color from b_0. Since b got a new Γ-color and $b >_{\mathbf{P}} b_0$ we conclude that b had to be played after k_1, hence after $B_1, \ldots, B_s$. This with (**g**) gives us $b_s{\downarrow_{\mathbf{Q}}} \subseteq b{\downarrow_{\mathbf{Q}}}$ and we get $b >_{\mathbf{Q}} g(b_s)$. Now, b can be matched with $g(b_s)$ and q_s can be matched with b_0. This new matching witnesses $\text{width}(A \cup B) = |A|$ in $\mathbf{Q}$.

To verify the rest of the properties of $\mathbf{Q}$ and Δ we use arguments similar to the previous cases. This completes our proof of Property (**l**).

□

The following fact is an useful corollary from (**l**).

COROLLARY 3.15. Colors from $\Gamma(A)$ are reused only in $B_0 \cup K_1$, colors from $\Gamma(B_j)$ $(1 \leqslant j \leqslant k)$ are reused only in $W_j \cup K_{j+1}$, and colors from $\Gamma(B_k)$ are reused only in W_k. In particular

$$\begin{aligned} f &= b_0 + k_1, \\ f_j &= w_j + k_{j+1}, \text{ for } j = 1, \dots, k-1, \\ f_k &= w_k. \end{aligned}$$

PROOF. For $j > 1$ and $k \in K_j$ by (**j**) we know that k_j has no available Γ-tops from B_j. The next best (with respect to $\prec$) candidates for Γ-color for k are Γ-tops lying in B_{j-1} at the moment when k arrives. By (**l**) there always exists some Γ-top in B_{j-1} that can be used to color $k \in K_j$, hence, $\Gamma(K_j) \subseteq \Gamma(B_{j-1})$. For $k \in K_1$ we deduce by (**j**) and $A < k$ that the only Γ-colors available for k are those with Γ-tops in A at the moment when k arrives. By (**c**) there is always Γ-top available in A.

We just argued that K_1 takes Γ-colors from A and K_j $(j > 1)$ takes Γ-colors from B_{j-1}. Since we already established that the W_j's take colors from B_j's and B_0 from the corresponding A we are done. □

OBSERVATION 3.16.

$$b_1 + \ldots + b_j \leqslant b_0 + w_1 + \ldots + w_{j-1}, \qquad j = 1, \dots, k.$$

PROOF. Consider initial parts of the presentation $\mathbf{P}$: let $P^{[0]} = (A, B_0)$, $P^{[j]} = (A, B_0, B_1, \dots, B_j, C_j)$ for $j \geqslant 1$. Let $F^{[j]}$ be the set of points from A whose Γ-colors are reused by some points in $P^{[j]} \setminus A$ and let $f^{[j]} = \left|F^{[j]}\right|$. Since B_j $(j > 1)$ is played right after $P^{[j-1]}$ and all points from B_j gets a new Γ-color it follows that $B_j{\downarrow} \subseteq F^{[j-1]}$. On the other hand from $|A| = \text{width}(\mathbf{P})$ (by (**d**)) and $B_1{\downarrow} \subseteq B_2{\downarrow} \subseteq \ldots \subseteq B_{k+1}{\downarrow}$ (by (**g**)) we get that $|B_1| + |B_2| + \ldots + |B_j| \leqslant |B_j{\downarrow}|$. The combination of these facts yields:

$$b_1 + b_2 + \ldots + b_j \leqslant |B_j{\downarrow}| \leqslant f^{[j-1]}, \text{ for } j = 1, \dots, k.$$

Let $b_i^{[j]}$, $w_i^{[j]}$ and $k_i^{[j]}$ be the respective counting variables. Note that for $i \leqslant j$ we have $b_i^{[j]} = b_i$ (all points from B_i are already introduced in $P^{[j]}$) and $w_i^{[j]} = w_i$ (by (**i**) points from W_i are introduced at the beginning of C_i). Therefore we will omit the notational overhead at the b's and w's.

Corollary 3.15 obviously induces on our initial part $P^{[j]}$ the following

$$\begin{aligned} f^{[j]} &= b_0 + k_1^{[j]}, \\ f_i^{[j]} &= w_i + k_{i+1}^{[j]}, \text{ for } j \geqslant i. \end{aligned}$$

Now, note that $k_i^{[j]} = \left|K_i^{[j]}\right| \leqslant \left|K_i^{[j]}{\downarrow} \cap B_i\right|$ by Property (**k**), as otherwise $K_i^{[j]} \cup (B_i - B_i \cap K_i^{[j]}{\downarrow})$ would form a larger antichain than B_i itself. By (**j**), points from K_i dominate only such points in B_i which Γ-colors are already used on C, in particular $K_i^{[j]}{\downarrow} \cap B_i \subseteq F_i^{[j]}$. Putting this together we get

$$k_i^{[j]} = \left|K_i^{[j]}\right| \leqslant \left|K_i^{[j]}{\downarrow} \cap B_i\right| \leqslant \left|F_i^{[j]}\right| = f_i^{[j]}.$$

Combining few last displays and using the fact that $k_{j-1}^{[j-1]} = 0$ we obtain our Observation:

$$\begin{aligned} b_1 + \ldots + b_j &\leqslant f^{[j-1]} \\ &= b_0 + k_1^{[j-1]} \\ &\leqslant b_0 + f_1^{[j-1]} \\ &= b_0 + w_1 + k_2^{[j-1]} \\ &\ldots \\ &\leqslant b_0 + w_1 + \ldots + w_{j-1}. \end{aligned}$$

□

Property (**m**).

$$\begin{aligned} &b_j = w_j + k_j, \ \textit{for} \ \ j = 1, \ldots k, \\ &w \leqslant f + b_0. \end{aligned}$$

Proof. Property (**k**) implies $b_j \geqslant w_j + k_j$. Suppose $b_j > w_j + k_j$. Consider **Q** which is obtained from **P** by adding a new point p which goes into W_j. Point p is played at the beginning of C_j with $p{\downarrow} = A \cup B_1 \cup \ldots \cup B_j$.

By (**l**) we know that some Γ-top, say of color γ, remains in B_j in **P**, i.e., it is not used on level C. It is easy to see that coloring

$$\Delta(q) = \begin{cases} \gamma, & q = p, \\ \Gamma(q), & \text{otherwise} \end{cases}$$

is natural on **Q**.

As we added a new point, we need an argument that the width of $B_j \cup U_j$ does not increase in **Q**. This holds, as $b_j > w_j + k_j$ in **P** and $p > B_j$ in **Q**. Iterating this process we will finally reach $b_j = w_j + k_j$.

To prove the second inequality note first that

$$\begin{aligned} |B| &= |B_1 \cup \ldots \cup B_{k+1}| + |B_0| \\ &\leqslant |(B_1 \cup \ldots \cup B_{k+1}){\downarrow}| + |B_0| && \text{by } (\mathbf{d}) \\ &= |B_{k+1}{\downarrow}| + |B_0| && \text{by } (\mathbf{g}) \\ &\leqslant |F| + |B_0| \end{aligned}$$

where the last inequality follows from the fact that points in B_{k+1} receive new Γ-colors and therefore there is no Γ-top in their downsets.

Now assume that $w > f + b_0$. We construct a poset $\mathbf{Q}$ again by adding a new point p. This time p goes into B_0: $p{\downarrow}_{\mathbf{Q}} = A$ and $p{\uparrow}_{\mathbf{Q}} = \emptyset$ and p is presented together with other points from B_0. By (**c**) we can pick Γ-color used only in A, say γ, and again it is easy to verify that

$$\Delta(q) = \begin{cases} \gamma, & q = p, \\ \Gamma(q), & \text{otherwise} \end{cases}$$

is natural on $\mathbf{Q}$. The width of $\mathbf{Q}$ is the same as the width of $\mathbf{P}$ since $w > f + b_0 \geqslant |B|$ and $p \geqslant A$. Again we may iterate this construction until our inequality is satisfied. □

3.2.5. The upper bound: Calculations. We are ready to state the constraints for the linear program whose solution will prove that the strategy of Spoiler presented in Subsection 3.2.3 is indeed the best possible.

THEOREM 3.17. *If* $\mathbf{P}$ *is an up-growing semi-order and* Γ *is a natural coloring of* $\mathbf{P}$ *using* val-semi-upg(w) *colors and satisfying properties* (**a**)–(**m**), *then*

$$(\star)\begin{cases} b_1 + \ldots + b_i \leqslant b_0 + w_1 + \ldots + w_{i-1}, & i = 1, \ldots, k, \\ b_i - 2w_i \leqslant b_{i+1} - w_{i+1}, & i = 1, \ldots, k-1, \\ b_k \leqslant 2w_k, & \\ w - 2b_0 \leqslant b_1 - w_1. & \end{cases}$$

PROOF. The first set of inequalities was subject to Observation 3.16.

For the second set of inequalities recall first that $b_i = w_i + k_i$ (by (**m**)). This, together with Corollaries 3.14 and 3.15, gives $b_i = w_i + k_i \leqslant w_i + f_i = 2w_i + k_{i+1} = 2w_i + b_{i+1} - w_{i+1}$ which is just a rearrangement.

The proof of the third is much as of the previous one: $b_k = w_k + k_k \leqslant w_k + f_k = 2w_k$.

For the last inequality start with $w - 2b_0 \leqslant f - b_0$ from (**m**), replace f and then k_1 using Corollary 3.15 and again (**m**). □

CLAIM 3.18. From the system $(\star)$ of inequalities it follows that

$$\frac{w - b_0}{w} \leqslant \frac{F_{2k+2}}{F_{2k+3}}.$$

PROOF. Substituting $\alpha_i = b_i - w_i$ for $i = 1, \ldots, k$, let $\alpha_0 = w - b_0$ and $\alpha_{k+1} = 0$ in $(\star)$ we obtain

$$\begin{cases} b_i \leqslant w - \sum_{j=0}^{i-1} \alpha_j, & i = 1, \ldots, k, \\ 2\alpha_i \leqslant b_i + \alpha_{i+1}, & i = 1, \ldots, k, \\ 2\alpha_0 \leqslant w + \alpha_1. & \end{cases}$$

Eliminate b_i from the second line of inequalities by using the first one:

$$\alpha_i + \sum_{j=0}^{i} \alpha_j \leqslant w + \alpha_{i+1}, \quad i = 0, \dots, k. \tag{16}$$

Summing up of the inequalities from (16) with weights $F_{2(k-i)+1}$ yields:

$$\begin{aligned} &\sum_{i=0}^{k} \alpha_i F_{2(k-i)+1} + \sum_{i=0}^{k} \sum_{j=0}^{i} \alpha_j F_{2(k-i)+1} \leqslant \\ &\leqslant w \sum_{i=0}^{k} F_{2(k-i)+1} + \sum_{i=0}^{k} \alpha_{i+1} F_{2(k-i)+1}. \end{aligned} \tag{17}$$

Recall from (2) that $\sum_{i=0}^{k} F_{2(k-i)+1} = F_{2k+2}$ and hence

$$\sum_{i=0}^{k} \sum_{j=0}^{i} \alpha_j F_{2(k-i)+1} = \sum_{j=0}^{k} \alpha_j \sum_{i=j}^{k} F_{2(k-i)+1} = \sum_{i=0}^{k} \alpha_i F_{2(k-i)+2}.$$

The inequality (17) can be now rewritten into

$$\sum_{i=0}^{k} \alpha_i F_{2(k-i)+1} + \sum_{i=0}^{k} \alpha_i F_{2(k-i)+2} \leqslant w F_{2k+2} + \sum_{i=1}^{k+1} \alpha_i F_{2(k-i)+3}.$$

Thus $\alpha_0 F_{2k+3} = \alpha_0(F_{2k+1} + F_{2k+2}) \leqslant w F_{2k+2}$. This in turn can be written as:

$$\frac{w - b_0}{w} \leqslant \frac{F_{2k+2}}{F_{2k+3}}.$$

□

THEOREM 3.19. *The value of on-line chain partitioning game for up-growing semi-orders of width w does not exceed $\lfloor \frac{1+\sqrt{5}}{2} w \rfloor$.*

PROOF. For $w \leqslant 4$ with first few Properties, as we discussed on page 38, it is easy to verify that no more than $\lfloor \frac{3}{2} w \rfloor = \lfloor \frac{1+\sqrt{5}}{2} \rfloor$ colors may be forced. For $w \geqslant 5$ consider a semi-order $\mathbf{P}$ and its natural coloring Γ such that $|\Gamma(\mathbf{P})| = \text{val-semi-upg}(w)$. We may assume that pair $(\mathbf{P}, \Gamma)$ satisfies Properties (**a**)–(**m**). From the definition of the B_i's and the fact that no new Γ-color is used on C (by Corollary 3.8) it follows that

$$\text{val-semi-upg}(w) = |\Gamma(\mathbf{P})| = |A| + |B_1| + \ldots + |B_{k+1}|.$$

Now using the fact that A witnesses the width of $\mathbf{P}$ (by (**d**)) we get $|B_1| + \ldots + |B_{k+1}| \leqslant |A| = w$ and therefore $\text{val-semi-upg}(w) \leqslant 2w - b_0$.

We know that $(\star)$ holds for $\mathbf{P}$. Therefore we can use Claim 3.18 to get

$$\frac{\text{val-semi-upg}(w)}{w} = 1 + \frac{w - b_0}{w} \leqslant \frac{F_{2k+3} + F_{2k+2}}{F_{2k+3}} \leqslant \frac{F_{2k+4}}{F_{2k+3}} \leqslant \frac{1 + \sqrt{5}}{2}.$$

The last inequality is due to the fact that the sequence $(\frac{F_{2k+2}}{F_{2k+1}})_{k\geqslant 0}$ is monotone increasing with limit $\varphi = \frac{1+\sqrt{5}}{2}$. This, together with the fact that val-semi-upg(w) is an integer, completes the proof. $\square$

CHAPTER 4

Semi-orders with representation

In this Chapter we discuss an open problem left in the considered area. The value of the on-line chain partitioning game for semi-orders with representation is still unknown. On the other hand the value of the up-growing variant with representation comes immediately from Theorem 1.12 and it is w as Spoiler is unable to cheat Algorithm.

There are only, rather straightforward bounds for the general, not necessarily up-growing, case with representation. Probably, the first statement of this problem comes from Chrobak and Ślusarek [**CS88**]. We consider here a game in which:

- Spoiler presents unit-length intervals, one at time,
- Algorithm colors these intervals in such a way that intersecting ones have different colors.

This appears to be a quite natural question and since it is set completely in terms of intervals and their mutual intersections, it may be analyzed without using the language of posets. The value of this coloring game, denoted by val-semi-repr(w) for unit-length interval collections of clique-size at most w, is the largest integer n such that Spoiler can force Algorithm to use n colors. Equivalently val-semi-repr(w) is the least n such that Algorithm has a strategy using no more than n colors on unit-length interval collection of clique-size at most w.

FACT 4.1.

$$\left\lfloor \frac{3}{2} w \right\rfloor \leqslant \text{val-semi-repr}(w) \leqslant 2w - 1.$$

PROOF. The upper bound is given by any greedy algorithm (see Fact 2.3 in Chapter 2 for details). In order to prove the lower bound we present Spoiler's strategy forcing $3k$ colors on a collection of clique-size $2k$. This strategy looks as follows:

(i) Present a clique A of k identical, unit-length intervals. Let $1, \ldots, k$ be the colors used by Algorithm on them.

(ii) Present a clique B of $2k$ unit-length intervals all of which left endpoints lie in an unit slot to the right of A in such a way that Algorithm is finally forced to use new colors (not $1, \ldots, k$) on most to the left intervals from B.

In order to do that present first interval from B for example $\frac{1}{2}$ to the right of the right end of A. Then present $(i+1)$-th interval slightly to the right of already presented intervals in B with new colors (if there are such) and slightly to the left of intervals in B with old colors (if there are such) – see Figure 4.1. Doing that no matter what color Algorithm uses on introduced interval we keep the property that all intervals in B with a new color are still to the left of all intervals in B with an old color.

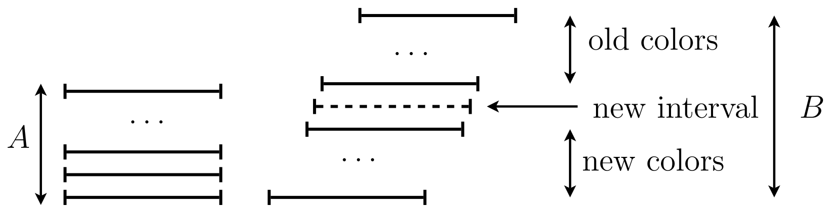

Figure 4.1. The construction of B separating intervals with new colors from those with old colors

(iii) Since B has $2k$ intervals and there are at most k old colors used on B, there must be at least k intervals in B with new colors. We secured that these new colors appeared on intervals that are closer to A than the others. Now, present k identical, unit-length intervals intersecting A and exactly k leftmost intervals in B (see Figure 4.2).

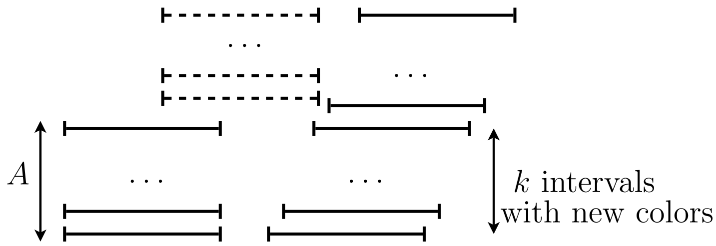

Figure 4.2. Last k intervals forcing Algorithm to use k brand new colors

It is clear that at least $3k$ colors will be used and the maximal clique has size $2k$. □

By a collection of proper intervals we mean one in which no interval is contained in the interior of any other. Obviously collections of unit-length intervals are proper. Moreover, by a straightforward argument one can locally rescale intervals, one by one, to show that **P** is a semi-order if and only if **P** has a proper representation. On the other hand presenting proper intervals instead of unit-length intervals may make a difference in on-line games. Introducing proper intervals, Spoiler provides less information to Algorithm and therefore he may force more colors than providing unit-length intervals. In particular, we slightly improve the lower bound for Spoiler presenting proper intervals from $\lfloor\frac{3}{2}w\rfloor$ to $\lceil\frac{3}{2}w\rceil$.

We believe that strategies forcing only $\lfloor\frac{3}{2}w\rfloor$ colors do not make use of the full Spoiler's power. In particular this feeling is based on the fact that the upper bound of $2w-1$ has been proved (by Chrobak and Ślusarek) to be sharp for all greedy algorithms and, as we will see in Fact 4.3, it is also sharp on a quite wider class of algorithms.

FACT 4.2 (Chrobak, Ślusarek [**CS88**]). There is a strategy for Spoiler in the on-line coloring game for unit-length intervals forcing any greedy algorithm to use $2w-1$ colors on collection of clique-size w.

PROOF. We present announced strategy assuming that Algorithm plays greedily. First, Spoiler introduces intervals a_1,b_1,...,a_{w-1},b_{w-1},a_w in such a way that the a_i's form a clique, the b_i's form a clique, all the a_i's are to the left of the b_i's and their left endpoints are sorted by: $l_{a_1} < l_{a_2} < \ldots < l_{a_w} < l_{b_1} < l_{b_2} < \ldots < l_{b_{w-1}}$ (see Figure 4.3) where l_x denotes the left endpoint of x. For each i, any greedy Algorithm

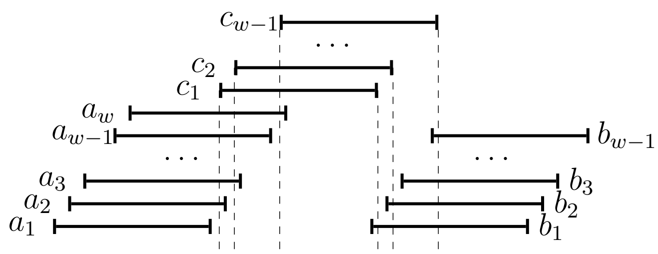

FIGURE 4.3. Spoiler's strategy forcing $2w-1$ colors on greedy algorithms

colors a_i and b_i with color i. Now, Spoiler presents a clique $c_1,\ldots,c_{w-1}$ in such a way that c_i intersects $a_{i+1},\ldots,a_w$ and $b_1,\ldots,b_i$ (see Figure 4.3). This forces Algorithm to use on each c_i a new color, reaching in total $2w-1$ colors. Clearly, the clique-size of presented collection is w. Our construction can be realized on-line by unit-length intervals e.g. by setting left ends of a_i, b_i and c_i to $\frac{i-1}{w}$, $2+\frac{i-1}{w}$ and $1+\frac{i-\frac{1}{2}}{w}$, respectively. □

From Fact 4.1 we know that if Algorithm plays well enough he can never be forced to use more than $2w-1$ colors on an unit-length interval collection of clique-size w. Actually, if after arrival of a new unit-length interval the clique-size is k then this new interval intersects at most $2(k-1)$ old ones, see Figure 4.4, and it can be colored by one of the $2k-1$ colors. This suggests that a smart local (in time) behavior of an algorithm may rely on not using more than $2k-1$ colors as long as the interval collection has current clique-size k. Such algorithms we will call locally smart.

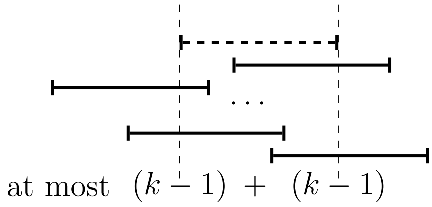

FIGURE 4.4. Intervals intersecting the dotted one must contain its left end or right end

Unfortunately in Fact 4.3 we will see that these algorithms are not much smarter than greedy ones. Actually they are finally forced to use $2w-1$ colors on a unit-length interval collection of clique-size w. Therefore to reduce the final upper bound of $2w-1$ one needs an algorithm that allows itself to be beaten during the game, i.e. to sacrifice the $2k$-th color on a current collection of clique-size k.

FACT 4.3. There is a strategy for Spoiler in the on-line coloring game for unit-length intervals forcing any locally smart algorithm to use $2w-1$ colors on collection of clique-size w.

PROOF. The strategy proving the fact is provided by w phases:
Phase 1. Present w non-intersecting intervals and, when reading from the left denote them by $a_1^1, \ldots, a_w^1$. Locally smart algorithm has to use the same color on all of them, say color 1.
Phase 2. Present $2(w-1)$ intervals $b_1^2, c_1^2, \ldots, b_{w-1}^2, c_{w-1}^2$ in such a way that (see Figure 4.5):

(i) the left endpoint of b_i^2 lies in a_i^1 and the right endpoint of b_i^2 lies in the space between a_i^1 and a_{i+1}^1,
(ii) the left endpoint of c_i^2 lies in the space between a_i^1 and a_{i+1}^1 and the right endpoint of c_i^2 lies in a_{i+1}^1,
(iii) b_i^2 intersects c_i^2 and there is no other intersection between intervals from Phase 2, i.e., c_i^2 does not intersect b_{i+1}^2.

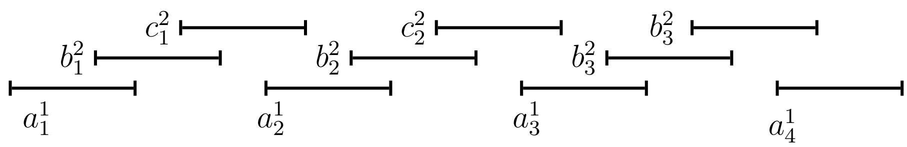

FIGURE 4.5. Forcing construction for $w=4$, Phase 2

Observe that every interval from Phase 2 intersects some a_i^1. Since the clique-size after Phase 2 is 2 any locally smart algorithm has only 2 new colors available so that it colors all presented intervals with say 2 or 3. Thus, for all i's the set of colors of $\{b_i^2, c_i^2\}$ is $\{2,3\}$.

Phase j. (for $2 < j \leqslant w$) Present $2(w-j+1)$ intervals $b_1^j, c_1^j, \dots, b_{w-j+1}^j$, c_{w-j+1}^j in such a way that:

(i) the left endpoint of b_i^j lies in the intersection of b_i^{j-1} and c_i^{j-1}, the right endpoint of b_i^j lies in the space between c_i^{j-1} and b_{i+1}^{j-1},

(ii) the left endpoint of c_i^j lies in the space between c_i^{j-1} and b_{i+1}^{j-1}, the right endpoint of c_i^j lies in the intersection of b_{i+1}^{j-1} and c_{i+1}^{j-1},

(iii) b_i^j intersects c_i^j and there is no other intersection between intervals from Phase j, i.e. c_i^j does not intersect b_{i+1}^j.

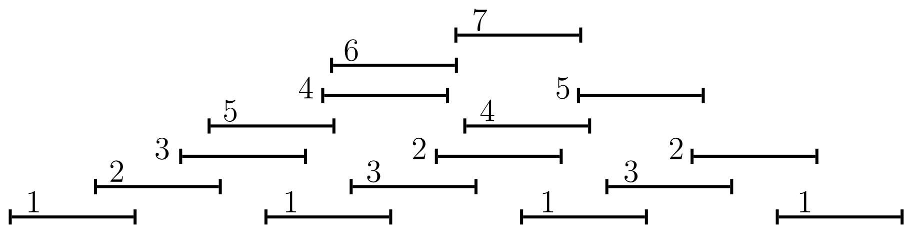

FIGURE 4.6. Forcing construction for $w = 4$

First of all, we induct on j to show that after Phase j the clique-size is j. Indeed, for $j > 2$ any intersection of intervals from Phase j is of the form $b_i^j \cap c_i^j$, and this intersection does not meet intervals from Phase $j-1$. Thus, a clique containing two intervals from Phase j may have at most $2 + (j-2)$ intervals (the clique-size after Phase $j-2$ is inductively $j-2$). Obviously, a clique with at most one interval from Phase j has at most $1 + (j-1)$ intervals, which ends our inductive argument.

Now, analyzing again all intersections from Phase j, we see that $b_i^j \cap c_i^j$ is in fact nested in the space between c_i^{j-1} and b_{i+1}^{j-1}. Moreover, the space between c_i^j and b_{i+1}^j is contained in $b_{i+1}^{j-1} \cap c_{i+1}^{j-1}$. In our construction every interval from Phase j has one endpoint in the intersection of intervals from previous phase and other one in the space between such intervals. This gives us that one endpoint always lies in the intersections of intervals from Phases:

$$j-1, j-3, \dots,$$

while the other one must lie in the intersections of intervals from Phases:

$$j-2, j-4, \dots.$$

Recalling that $\{b_i^2, c_i^2\}$ is colored by $\{2, 3\}$ we can keep inductively that $\left\{b_i^k, c_i^k\right\}$ is colored by $\{2k-2, 2k-1\}$ for all $k < j$. This means that all colors used on previous phases are not available for Phase j. Thus, locally smart algorithm is forced to use new colors $2j-2$ and $2j-1$ on $\left\{b_i^j, c_i^j\right\}$. Setting $j=w$ we see that $2w-1$ colors were forced to be used.

Our construction can be realized on-line by unit-length intervals e.g. by setting left ends of a_i^1, b_i^j and c_i^j to $2(i-1)$, $2(i-1)+(j-2)+\sum_{k=1}^{j-2}\frac{1}{3^k}+\frac{2}{3^{j-1}}$ and $2(i-1)+(j-1)+\sum_{k=1}^{j-2}\frac{1}{3^k}+\frac{1}{3^{j-1}}$, respectively. $\square$

Bibliography

[BBM07] Patrick Baier, Bartłomiej Bosek, and Piotr Micek. On-line chain partitioning of up-growing interval orders. *Order*, 24(1):1–13, 2007.

[Bea76] Dwight R. Bean. Effective coloration. *J. Symbolic Logic*, 41(2):469–480, 1976.

[Bos08] Bartłomiej Bosek. *On-line chain partitioning as approach to scheduling.* PhD thesis, Jagiellonian University, 2008.

[Bro05] Przemysław Broniek. On-line chain partitioning as a model for real-time scheduling. In *Proceedings of the Second Workshop on Computational Logic and Applications (CLA 2004)*, volume 140 of *Electron. Notes Theor. Comput. Sci.*, pages 15–29 (electronic), Amsterdam, 2005. Elsevier.

[CS88] Marek Chrobak and Maciej Ślusarek. On some packing problem related to dynamic storage allocation. *RAIRO Inform. Théor. Appl.*, 22(4):487–499, 1988.

[Dil50] Robert P. Dilworth. A decomposition theorem for partially ordered sets. *Ann. of Math. (2)*, 51:161–166, 1950.

[Fel93] Stefan Felsner. *Interval orders: Combinatorial Structure and Algorithms.* PhD thesis, Technische Universität Berlin, 1993. http://www.math.tu-berlin.de/~felsner/Paper/diss.pdf.

[Fel97] Stefan Felsner. On-line chain partitions of orders. *Theoret. Comput. Sci.*, 175(2):283–292, 1997. Orders, algorithms and applications (Lyon, 1994).

[Fis70] Peter C. Fishburn. Intransitive indifference with unequal indifference intervals. *J. Mathematical Psychology*, 7:144–149, 1970.

[GGL95] Ronald L. Graham, Martin Grötschel, and László Lovász, editors. *Handbook of combinatorics. Vol. 1, 2.* Elsevier Science B.V., Amsterdam, 1995.

[Kie81a] Henry A. Kierstead. An effective version of Dilworth's theorem. *Trans. Amer. Math. Soc.*, 268(1):63–77, 1981.

[Kie81b] Henry A. Kierstead. Recursive colorings of highly recursive graphs. *Canad. J. Math.*, 33(6):1279–1290, 1981.

[Kie83] Henry A. Kierstead. An effective version of Hall's theorem. *Proc. Amer. Math. Soc.*, 88(1):124–128, 1983.

[Kie86] Henry A. Kierstead. Recursive ordered sets. In *Combinatorics and ordered sets (Arcata, Calif., 1985)*, volume 57 of *Contemp. Math.*, pages 75–102. Amer. Math. Soc., Providence, RI, 1986.

[KMT84] Henry A. Kierstead, George F. McNulty, and William T. Trotter, Jr. A theory of recursive dimension for ordered sets. *Order*, 1(1):67–82, 1984.

[KT81] Henry A. Kierstead and William T. Trotter, Jr. An extremal problem in recursive combinatorics. In *Proceedings of the Twelfth Southeastern Conference on Combinatorics, Graph Theory and Computing, Vol. II (Baton Rouge, La., 1981)*, volume 33, pages 143–153, 1981.

[Sch80] James H. Schmerl. Recursive colorings of graphs. *Canad. J. Math.*, 32(4):821–830, 1980.

[Sch82] James H. Schmerl. The effective version of Brooks' theorem. *Canad. J. Math.*, 34(5):1036–1046, 1982.

[SS58] Dana Scott and Patrick Suppes. Foundational aspects of theories of measurement. *J. Symb. Logic*, 23:113–128, 1958.

[Tro92] William T. Trotter. *Combinatorics and partially ordered sets*. Johns Hopkins Series in the Mathematical Sciences. Johns Hopkins University Press, Baltimore, MD, 1992. Dimension theory.