BEETHOVEN Classic 3

Polish-German funding initiative

Order & Geometry

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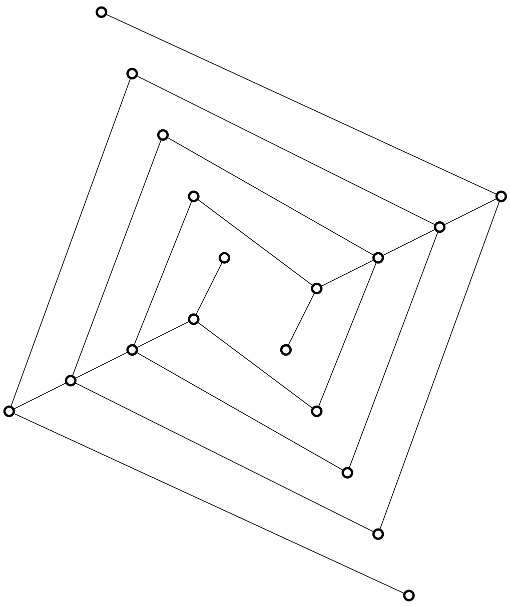
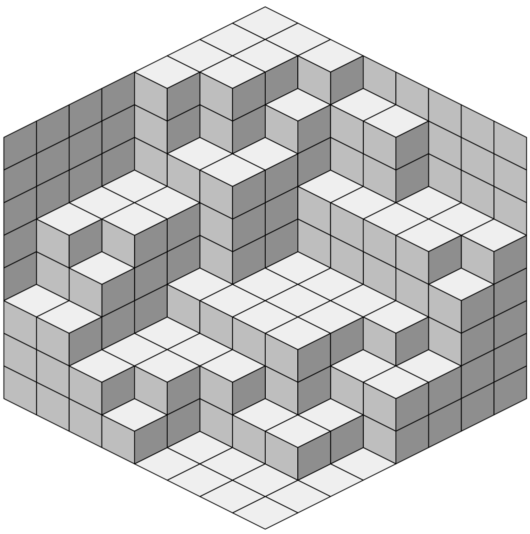
Host institution: UJ Kraków

PI Germany: Stefan Felsner
Host institution: TU Berlin

Berlin & Kraków, December 2018

Project duration: 3 years





I. CORE DATA

1. Title of the Research Project

Order & Geometry

2. Acronym

Twelve characters maximum, same as the ones given in the ZSUN/OSF and ELAN submission systems.

ORGEO

3. Name and affiliation of the Polish PI

Academic Title: Dr.

First Name: Piotr

Last Name: Micek

Host institution Main level:
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4. Name and affiliation of the German PI

Academic Title: Prof. Dr.

First Name: Stefan

Last Name: Felsner

Host institution Main level:
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Host institution Faculty level:
Fakultät II, Institut für Mathematik

Host institution Email address:
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5. Subject classification

In the case of an interdisciplinary project, please indicate the main discipline. Please refer to NCN panels and DFG subject areas.

Mathematics
For NCN: ST1 Mathematics, ST1_14 Combinatorics
For DFG: Research Area 33 (Mathematics) Fachkollegium 312

6. Keywords

Please submit at least one and at most six keywords separated by a semicolon

graph; poset; geometric representation; chi-boundedness; dimension

7. Project duration for which funding is requested

36 months

8. Summary

Research project objectives/research hypothesis; research project methodology; expected impact of the research project on the development of science; added value of bilateral cooperation; up to 1 page.

Graphs and orders defined by means of geometric objects provide a rich class of examples in combinatorics and graph theory. The geometric intuition often guides through constructions that are complex and complicated otherwise. Moreover, graphs and orders defined in terms of geometric objects model dependencies in optimization problems and theoretical computer science. Within this project we focus on the combinatorial side of this realm. The research is grouped into three lines and each line will be motivated by some notoriously open, long-standing problems such as: (1) What is the best possible bound for the chromatic number of intersection graphs of axis-aligned rectangles in the plane? (with essentially no progress since the seminal paper by Asplund and Grünbaum in 1960); (2) Is the queue number of planar graphs bounded? (conjectured by Heath, Leighton and Rosenberg in 1992); (3) Is the Boolean dimension of planar posets bounded? (posed by Nešetřil and Pudlák in 1989).

These problems exemplify different types of interplay between orders (or orderings) and geometry in combinatorics. The basic concept of our research is to understand and exploit these interplays.

II. RESEARCH TEAM

1. Research Team composition

Research Team members can be listed as Principal Investigators, Co-investigators, Post-docs, scholarship grantees or technical staff.
Principal Investigators and Co-Investigators need to provide their CVs to be attached in the appropriate sections of the ZSUN/OSF and ELAN system.
For Polish applicants: Please note that no personal data (names) for post-docs and scholarship grantees should be included.

German research team members:

Name and academic title	Career break	Host institution	Percentage share of the overall working time devoted to the project	Participant in another proposal within DFG-NCN call?
Stefan Felsner, Prof. Dr.	no	Technische Universität Berlin	25% (10h/week) (36 months)	no
postdoc	no	Technische Universität Berlin	100% (40h/week) (12 months)	no
PhD student	no	Technische Universität Berlin	100% (40h/week) (36 months)	no

Further PhD students will also contribute and participate in research and exchange related to the project.

Polish research team members:

Principal Investigator, Co-investigators, Post- docs, scholarship grantees or technical staff.

Name and academic title	Career break	Host institution	Percentage share of the overall working time devoted to the project	Participant in another proposal within DFG-NCN call?
Piotr Micek, Dr.	no	Jagiellonian University	40% (16h/week) (36 months)	no
postdoc	no	Jagiellonian University	100% (40h/week) (12 months)	no
PhD student	no	Jagiellonian University	100% (40h/week) (36 months)	no
MSc student	no	Jagiellonian University	40% (16h/week) (36 months)	no
MSc student	no	Jagiellonian University	40% (16h/week) (36 months)	no

2. Cooperation Partners

only persons who do not seek funding from NCN/DFG in this call; no CVs necessary

Name	Affiliation	Percentage share of the overall working time devoted to the project	Source of funding
Jean Cardinal	Université Libre de Bruxelles	3% (1h/week)	own
Daniel Gonçalves	CNRS & Université de Montpellier	3%	own
Gwenaël Joret	Université Libre de Bruxelles	6%	own
Kolja Knauer	Université Aix-Marseille	3%	own
Patrice Ossona de Mendez	EHESS, Paris	3%	own
William T. Trotter	Georgia Institute of Technology	6%	own
Torsten Ueckerdt	Karlsruher Institut für Technologie	3%	own
Bartosz Walczak	Uniwersytet Jagielloński, Kraków	3%	own

III. DESCRIPTION OF THE RESEARCH PROJECT

1. Current knowledge in the field and preliminary work

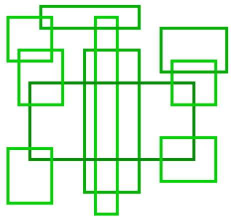
Graphs and orders defined by means of geometric objects provide a rich class of examples in combinatorics and graph theory. The geometric intuition often guides through constructions that are complex and complicated otherwise. Moreover, graphs and orders defined in terms of geometric objects model dependencies in optimization problems and theoretical computer science. Within this project we focus on the combinatorial side of this realm. The research is grouped into three lines and each line will be motivated by some notoriously open, long-standing problems.

The most commonly studied geometrically defined graphs are containment graphs, intersection graphs, and contact graphs. Containment graphs naturally come with a partial order relation but posets also capture relevant structural aspects in many types of intersection and contact graphs. We aim at exploiting these connections between geometrically defined graphs and orders. This is a rather broad program, in the project description we focus on three more specific problem areas where the interplay of order and geometry is relevant, these are:

- Chi-bounded classes of geometrically defined graphs
- Geometrically defined classes of graphs with linear structure
- Geometry and encodings

Chi–bounded classes of geometrically defined graphs

Recall that $\chi(G)$ is the chromatic number of graph G and $\omega(G)$ is the clique number of G that is the size of the largest clique in G. A class of graphs is *χ-bounded*, if there is a function $f : \mathbb{N} \to \mathbb{N}$ such that $\chi(G) \leqslant f(\omega(G))$ holds for any graph G from the class. Since there are triangle-free graphs with arbitrarily large chromatic number χ-boundedness is a non-trivial property. In fact, it constitutes a very lively field of research. E.g. Scott, Seymour et al. published a whole series of papers (see e.g. [CSSS17]) containing, in particular, proofs of three long-standing conjectures of Gyárfás [Gya87].



The study of χ-boundedness for geometric intersection graphs was initiated by Asplund and Grünbaum [AG60]. They proved that every family $\mathcal{F}$ of axis-aligned rectangles in the plane satisfies $\chi(\mathcal{F}) \leqslant 4\omega(\mathcal{F})^2 - 3\omega(\mathcal{F})$. The proof uses a partial order on crossing rectangles and a degeneracy argument. For general families of axis-aligned rectangles, we do not know much more than the result of Asplund and Grünbaum. The lower bound is still linear and the upper bound was only modestly improved to $\chi(\mathcal{F}) \leqslant 3\omega(\mathcal{F})^2 - 2\omega(\mathcal{F}) - 1$ by Hendler [Hen98]. It is a true challenge to verify whether

- $\chi(\mathcal{F}) = o(\omega^2(\mathcal{F}))$, for every axis-aligned family $\mathcal{F}$ of rectangles.

For families $\mathcal{F}$ of rectangles with no containment between rectangles, Chalermsook [Cha11] obtained $\chi(\mathcal{F}) = O(\omega(\mathcal{F}) \log \omega(\mathcal{F}))$. A linear bound for this specific case would also improve the quality of the best known approximation algorithm for the Maximum-Independent-Set-of-Rectangles problem (see [Cha11]).

Burling [Bur65] showed that triangle-free intersection graphs of axis-aligned boxes in $\mathbb{R}^3$ can have arbitrarily large chromatic number. The graphs used for the construction are now known as Burling graphs.

In the 1970s, Paul Erdős asked whether intersection graphs of line segments in the plane are χ-bounded. A negative answer was provided by Pawlik, Kozik, Krawczyk, Lasoń, Micek, Trotter, and Walczak [PKK+14]: The authors represented Burling graphs as intersection graphs of segments in the plane. This result also disproves the conjecture of Scott [Sco97] that, for every graph H, the class of graphs excluding every subdivision of H as an induced subgraph is χ-bounded. More recently Burling graphs have been used to disprove a conjecture about orthogonal tree-decompositions, see Dujmović et al. [DJM+18] and Felsner, Micek et al. [FJM+18]. Studying particular properties of Burling graphs is an exciting topic on its own and by now they have shown the potential to test long-standing conjectures in graph theory.

The construction from [PKK+14] was extended in [PKK+13] to other shapes like axis-aligned ellipses, rhombuses, L-shapes, etc. There is some evidence that unrestricted scaling in two directions is the key property, necessary to make the chromatic number large, while keeping the clique number small. For instance, Suk [Suk14] proved that for families $\mathcal{F}$ of *unit-length* segments in the plane $\chi(\mathcal{F})$ is bounded by a double exponential function of $\omega(\mathcal{F})$. Also families of curves attached to a single line (outerstrings) have χ bounded in terms of ω, see a paper by Micek et al. [LMPW14] and by Rok and Walczak [RW17].

Interestingly, the complements of intersection graphs of segments in the plane are χ-bounded. In that case, the chromatic number χ has been shown to be $O(\omega^4)$ by Pach, Törőcsik et al. [LJMT94]. The proof for this bound is a beautiful example of the interplay of orders and geometry: four partial orders are defined on the family of segments, a four times repeated application of Dilworth's Theorem then yields the result. Pach, Tardos and Tóth [PTT17] show that the same bound holds for the disjointness graph of segments in arbitrary dimension. Disjointness graphs of curves in the plane have also been studied, they are not χ-bounded in general but as soon as the curves are x-monotone and intersect the y-axis the precise χ-bounding function $\frac{\omega+1}{2}\binom{\omega+2}{3}$ has been established by Pach and Tomon [PT18].

Kim, Kostochka and Nakprasit [KKN04] showed that families $\mathcal{F}$, of homothetic copies of a fixed convex compact set in the plane, have $\chi(\mathcal{F}) \leqslant 6\omega(\mathcal{F})$. The result was generalized (with a very simple counting argument) to pseudo-discs in [MP13]. A family $\mathcal{F}$ of simply connected sets in the plane, is a family of *pseudo-discs*, if the boundaries of every two sets from $\mathcal{F}$ intersect in at most two points. Micek and Pinchasi show that $\chi(\mathcal{F}) \leqslant 19\omega(\mathcal{F})$ for families of pseudo-discs. It is an annoying open problem to improve this bound for at least a tiny bit.

Geometrically defined classes of graphs and linear patterns

Interval graphs may be considered to be the best understood class of graphs with linear structure. This class was introduced by Benzer in 1959 [Ben59] and helped to understand the linear structure of the DNA. A second classical example are permutation graphs which are readily described by a permutation (linear order) of the appropriately labelled vertices. Alternatively permutation graphs can be described as intersection graphs of segments with endpoints on two parallel lines. The modern theory of geometric intersection graphs was established in the 1990s by Kratochvíl [Kra91a, Kra91b] and Matoušek [Kra91a, Kra91b, KM91, KM94]. By now geometric intersection graphs are ubiquitous in discrete and computational geometry, and deep connections to other fields such as complexity theory [SSv03, Sch09, Mat14] and order dimension theory [CHO$^+$14, Fel14, CFHW18] have been established.

A family $\mathcal{F}$ of geometric objects is called *grounded* if every element of $\mathcal{F}$ is contained in a half-plane H and touches the boundary line ∂H of H in an *anchor point*. Listing the objects according to their anchor point gives a (canonical) linear ordering of the vertices. We have already mentioned intersection graphs of grounded strings (outer strings) in the context of χ-binding.

Outer segment graphs form a natural subclass of outer string graphs. They also generalize the class of circle graphs, which are intersection graphs of chords of a circle. Outerplanar graphs form a proper subclass of circle graphs [WP85], hence of outer segment graphs. Cabello and Jejčič [CJ17] proved that a graph is outerplanar if and only if its 1-subdivision is an outer segment graph. Intersection graphs of rays in two directions (a subclass of outer segment graphs, see [CFHW18]) have been studied by Soto and Telha [ST11], they show connections with the jump number of some orders and hitting sets of rectangles. The class has been further studied by Shrestha et al. [STU10], and Mustaţă et al. [MNT$^+$16].

Intersection graphs of L-shapes anchored at their bend has been investigated as *hook-graphs* and as *max point-tolerance graphs*, see [Hix13], [CCF$^+$17], and [ST15]. They generalize interval graphs and have various geometric representations and characterizations, e.g., they can be seen as intersection graphs of the rectangles spanned by the L's. A direct proof of χ-boundedness for this class (hopefully with a linear bound) would be of great interest. The recognition problem for hook-graphs and even for the still simpler intersection graphs of grounded vertical and horizontal segments (*stick graphs*) is also open. For the case where the order of anchor points on the grounding line is prescribed there is a polynomial recognition algorithm due to De Luca et al. [LHK+18].

Stick graphs and many related bipartite intersection graphs have been studied by Felsner, Hoffmann et al. [CFHW18]. The aim of that paper was to identify the inclusion order on the graph classes and the order dimension turned out to be a very effective tool. Cardinal, Felsner et al. [CFM+18] introduced the 'Cycle Lemma' which allows to prescribe the order of anchor points for certain graphs. The lemma was used in [CFM+18] and [JT18] to separate further grounded classes of graphs. Jelínek and Töpfer [JT18] also studied forbidden patterns. It had been observed by several groups that hook-graphs can be characterized by a forbidden pattern on the anchor sequence of four vertices, see Figure 1. Jelínek and Töpfer show that *grounded-L graphs*, i.e., graphs admitting an intersection representation by L's anchored with the upper end of the vertical bar at a horizontal line, admit a forbidden pattern characterization with two patterns on four vertices. The characterization of a class of graphs by forbidden vertex order patterns might conceivably lay the grounds for efficient recognition algorithms. Note, however, that a graph class characterized by a forbidden vertex order pattern may have NP-hard recognition [DGR95]. On the other hand Hell et al. [HMR14] unified many previous results by giving a general polynomial time recognition algorithm for all classes described by a set of forbidden patterns of order at most three.

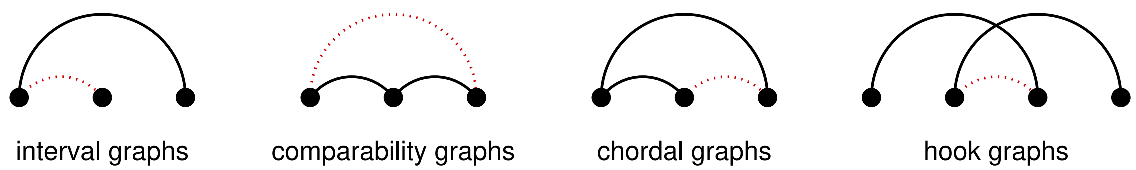


Figure 1: Forbidden order patterns for graph classes. Solid arcs denote compulsory edges and dotted arcs denote compulsory non-edges.

Forbidden patterns are also at the core of important parameters studied for graphs and posets. In a total order of the vertices of a graph, an independent pair of edges can be crossing, nested, or disjoint. A *k-stack* layout (respectively, *k-queue* layout) of a graph consists of a total order of the vertices, and a partition of the edges into k sets of pairwise non-crossing (respectively, non-nested) edges. Motivated by numerous applications, stack layouts (also called book embeddings) and queue layouts are widely studied, see e.g. a survey on stack and queue layouts by Dujmović and Wood [DW04] and a survey vertex orderings in a broader context by Diaz et al. [DPS02].

When the vertex order is fixed, the minimum number of queues required for a queue layout equals the maximum size of a nesting family of edges. Computing the minimum number of stacks, however, is NP-hard even when the vertex order is prescribed. Despite intense research on the parameters some of the problems and conjectures from the seminal paper by Heath, Lipton and Rosenberg [HLR92] are still unresolved. Two central questions in the field are:

- Is the queue-number of planar graphs bounded?
- Is the stack-number or the queue-number bounded by a function of the other?

Nowakowski and Parker [NP89b] defined the stack-number of a poset as the stack-number of its Hasse diagram viewed as a dag, i.e., the vertex ordering has to be a linear extension. They derive a general lower bound on the stack-number of a planar poset and an upper bound on the stack-number of a lattice. They conclude by asking

- whether the stack-number of the class of planar posets is unbounded.

There have been some recent attacks to the problem, see e.g. Frati et al. [FFR13], still the question in its general form remains open.

Heath and Pemmaraju [HP97] initiated the study of queue layouts of posets. Again, this is the queue-number of the diagram of the poset, whence the vertex ordering has to be a linear extension. They observe that the queue-number of the class of planar posets is unbounded, and bound the queue-number of a planar poset in terms of its width. They conjecture that a poset of width w has queue-number at most w. Knauer, Micek and Ueckerdt [KMU18] continue the study of the queue-number of posets. They have shown that a planar poset of width w has queue-number at most $3w-2$, while the bound on the queue-number for general posets remains $O(w^2)$.

Geometry and encodings

The most important measure for the complexity of a poset is its dimension. The *dimension* $\dim(P)$ of a poset P is the least integer d such that points of P can be embedded into $\mathbb{R}^d$ in such a way that $x \leqslant y$ in P if and only if the point of x is below the point of y with respect to the product order of $\mathbb{R}^d$. Though this definition justifies the geometric intuition behind the notion of dimension, usually we work with the following equivalent. A *realizer* of a poset P is a set $\{L_1,\ldots,L_d\}$ of linear extensions of P such that for every $x,y \in P$

$$x \leqslant y \text{ in } P \iff (x \leqslant y \text{ in } L_1) \wedge \cdots \wedge (x \leqslant y \text{ in } L_d),$$

and the dimension of P is the minimum size of its realizer.

This reveals the second nature of the dimension: Realizers provide a way to succinctly encode posets. Indeed if a poset is given with a realizer witnessing dimension d, then a query of the form "is $x \leqslant y$?" can be answered by looking at the relative position of x and y in each of the d linear extensions of the realizer. This application motivates the following more powerful encoding of posets proposed by Nešetřil and Pudlák [NP89a] in 1989. The *Boolean realizer* of a poset P is a set of permutations $\{L_1,\ldots,L_d\}$ of elements of P for which there exists a d-ary Boolean formula ϕ such that

$$x \leqslant y \text{ in } P \iff \phi((x \leqslant y \text{ in } L_1),\ldots,(x \leqslant y \text{ in } L_d)) = 1,$$

and the *Boolean dimension* of P, denoted $\mathrm{bdim}(P)$, is the minimum size of its Boolean realizer. Clearly, for every poset P we have $\mathrm{bdim}(P) \leqslant \dim(P)$.

The usual dimension of a poset on n elements may be linear in n. Nešetřil and Pudlák showed that Boolean dimension of posets on n elements is $O(\log n)$. They also provide an easy counting argument showing that there are posets on n elements with Boolean dimension at least $c \log n$ for some constant c.

A poset is *planar* if it has a planar diagram. Somewhat unexpectedly planar posets have arbitrarily large dimension. Kelly [Kel81] gave a construction that embeds the standard example S_n of an n-dimensional poset as a subposet into a planar poset (see Figure 2). This shows that the dimension of planar posets is unbounded. Still, the Boolean dimension of standard examples and Kelly's construction is at most 4. There is a beautiful open problem posed by Nešetřil and Pudlák in [NP89a] that remains a challenge with essentially no progress over the years:

- Is the Boolean dimension of planar posets bounded?

We believe to have made an important step towards a resolution of the problem by proving that posets with cover graphs of bounded treewidth have bounded Boolean dimension [FMM17]. This stays in contrast to the ordinary dimension as Kelly's examples have treewidth 3.

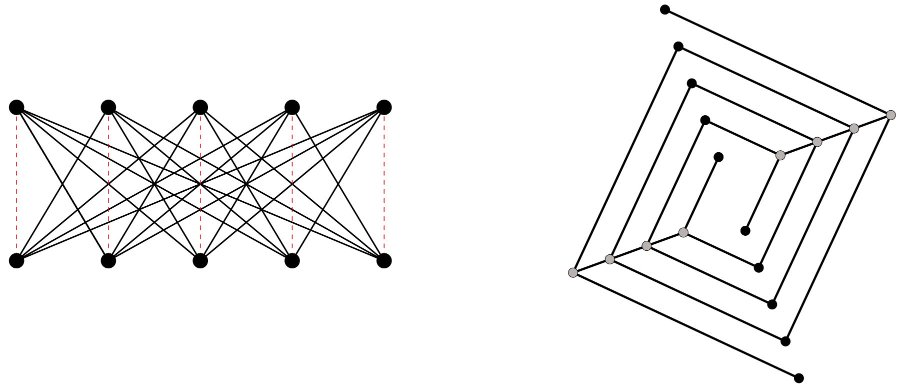


Figure 2: The standard example S_5 (left). Kelly's planar poset containing an induced S_5 (right).

Recently, Trotter and Walczak [TW17] studied the interplay of Boolean dimension with yet another concept, the *local dimension*. They propose constructions of families of posets where one of the parameters stays bounded while the other goes to infinity.

The usual dimension is known to be at most 3 for posets with cover graphs being forests (Trotter, Moore [TM77]) and at most 1276 for posets with cover graphs of tree-width 2 (Joret et al. [JMT+17]). As mentioned before, Kelly's examples have tree-width 3 and arbitrarily large dimension. This certifies that Boolean realizers are capable to represent natural classes of posets that are out of reach in the default setting.

It is tempting to speculate, whether the result from [FMM17] generalizes for broader classes of sparse posets. Besides planar posets, it might be true even for posets whose cover graphs exclude a fixed graph as a minor. On the other hand, we have an example (a subdivision of universal interval orders) that this result does not hold for posets whose cover graphs exclude a fixed topological minor. This line of research resembles the series of papers where poset dimension is bounded in terms of the height for posets whose cover graphs are planar (Streib and Trotter [ST14]), or have bounded tree-width (Micek et al. [JMM+16]), or exclude a fixed graph as a minor (Walczak [Wal17] and Micek, Wiechert [MW15], or belong to a fixed class with bounded expansion (Micek et al. [JMW17]).

By now Boolean dimension is not yet well understood. In particular we lack lower bound techniques. We even got stuck on the following easy looking question

- Is the Boolean dimension of a Boolean lattice of order n equal n?

If it is true we expect that there is a beautiful combinatorial argument behind it. See [KMM+18] for similar considerations concerning local dimension.

2. Objectives

In the preceding section we have reviewed some of the connections between order structures and geometric situations and the history of this research. Below we list some hard problems which serve as landmarks for our research. Impact on other problems can be expected even from partial solutions. To make progress on the problems we will have to combine the expertise in structural graph theory and extremal combinatorics from the Kraków side and the experience in working with geometrically defined graphs and orders from the Berlin side.

I. Chi-bounded classes of geometrically defined graphs.

I.1 Improve asymptotic bounds for the chromatic number of families of axis-aligned rectangles in terms of their clique-number, i.e. the upper bound $O(\omega^2)$ by Asplund and Grünbaum from 1960 and/or the trivial but still best known lower bound $O(\omega)$. Improve the bounds for special cases e.g. families of rectangles with no containment.

I.2 Improve the $O(\omega^4)$ bound for complements of intersection graphs of segments in the plane.

I.3 Study properties and geometric representations of Burling graphs.

II. Geometrically defined classes of graphs with linear structure.

II.1 Find a polynomial time recognition algorithm for stick graphs, i.e., grid intersection graphs with upper-left endpoint on the diagonal line. Or show that the problem is NP-complete.

II.2 Identify forbidden patterns of length four such that the corresponding class of graphs is easy to recognize.

II.3 Prove or disprove that the queue-number of planar graphs bounded. Improve the bounds for queue-numbers of planar posets. Prove or disprove that the stack-number of planar posets is bounded.

III. Geometry and encodings.

III.1 Solve or give some substantial partial solutions for a problem posed by Nešetřil and Pudlák (1989): Is the Boolean dimension of planar posets bounded?

III.2 Verify if the Boolean dimension of a Boolean lattice of order n is equal n?

3. Work Programme

Work Programme including proposed research methods, role of the participating research team members and added value of international cooperation

The nature of the proposed problems requires them to be tackled in cooperation of at least two team members: they are chosen at common borders of the scientific expertise of the involved research groups.

Our two teams have a long history of collaboration dating back to 2006 and including multiple shorter and long-term visits in both directions. In particular Piotr Micek was visiting the group of Felsner from 2013 to 2015 with Mobility Plus fellowship and in the fall of 2016 he was in Berlin as a substitute professor of Berlin Mathematical School. Stefan Felsner has been awarded a Alexander von Humboldt Polish Honorary Research Scholarship by the Foundation for Polish Science for a period of 5 months to be spent at Jagiellonian University. Piotr Micek and Stefan Felsner have jointly established the series of Order & Geometry workshops with events in 2013 (Berlin), 2016 (Gułtowy Palace), and 2018 (Ciążeń Palace). By now there are at least 15 joint research papers between the groups (just counting those with at least one of the principal investigators as a co-author). Many papers are published in the top journals in combinatorics (Combinatorica, SIAM Journal on Discrete Mathematics, Journal of Graph Theory, Discrete & Computational Geometry). Usually the Berlin team contributes the expertise in geometric aspects of representations of graphs and posets while the Kraków team accounts for its strong background in structural graph theory and algorithms. Undoubtedly, the present project will further strengthen the ties between the research groups from Berlin and Kraków.

The problems listed in the objectives section can serve for the doctoral students as entry points to their thesis projects. All (MSc-, PhD-) students within the project will be supervised jointly by both principal investigators. The PhD students are expected to spend at least three months at the respective collaborating institution. Additionally, some less ambitious projects can be offered to students preparing MSc theses in each of the collaborating teams. We also plan to offer exchange possibilities to MSc students. The postdoctoral researcher is to help getting momentum for the project in the first two years. Ideally the postdoctoral researcher should spent one year in Berlin and one year in Kraków.

We expect that the the Order & Geometry workshop in the Fall of 2020 will have major impact on the project . The workshop will be a reunion of the cooperation partners and additional experts in the field. Focusing the workshop on the problems of the project should lead to substantial progress. Previous workshops with this title in 2013, 2016, and 2018 have been very successful, e.g. the workshop of 2016 layed ground for research that directly lead to at least 5 publications, moreover, the fresh interest for local dimension and Boolean dimension of posets was the result of a very exciting final problem session at the workshop.

In the following sections we discuss our research plan in each of the three highlighted directions. We have some preliminary results which may serve as a nucleus for the research, real progress, however, has to be generated from new and creative ideas.

I. Chi-bounded classes of graphs.

Improving the upper bound for the chromatic number of families of axis-aligned rectangles in the plane in terms of their clique-number is one of the highlights of the project. We expect this to be difficult. So far there is no idea working for general families of rectangles different from the original Asplund and Grünbaum approach [AG60]. The best known lower bound, i.e. 3ω, is due to Kostochka but he never published his argument. We were able to reprove the same bound using Burling graphs. This might be a good starting point for more sophisticated constructions. From the upper bound side we have an idea to try to apply the iterative breadth-first search (BFS) technique that was proved to be very useful in arguments bounding the chromatic number (especially when some induced subgraph is forbidden). In short, the idea is to launch BFS on a given intersection graph and divide vertices into layers. Now the folklore result is that when the chromatic number of each layer is at most d, then the chromatic number of the whole graph is at most $2d$. Iterating BFS and narrowing to sublayers we finally may be able to use some geometric properties of our graphs to bound the chromatic number.

Another approach is to start with a result by Chalermsook, in [Cha11]. He proved a bound which applies to general families of rectangles. This bound is of order $O(\omega\gamma\log\omega)$ with γ being a rather un-intuitive parameter $1 \leqslant \gamma \leqslant \omega$. Simplifications of the algorithm may lead to better bounds for γ. As for simplifications, we plan to find simpler $O(\omega\log\omega)$ (or ideally $O(\omega)$) algorithms in special situations, e.g., when all rectangles are stabbed by a line, or even more restricted, when a line contains all upper-left corners of rectangles in the family.

A more general problem is to get a better understanding of properties that imply χ-boundedness. Families of pseudo-discs are χ-bounded and there are families of nice $\leqslant$4-intersecting objects that are unbounded. However, axis aligned rectangles are $\leqslant$4-intersecting and again χ-bounded. Is there a definition of pseudo-rectangles which implies χ-boundedness?

The $O(\omega^4)$ bound for the complements of intersection graphs of segments in the plane was established by Pach, Törőcsik et al. [LJMT94] using four partial orders and four applications of Dilworth's Theorem. In a very recent paper Pach and Tomon [PT18] obtain the same bound for disjointness graphs of x-monotone curves. Moreover they provide a lower bound of $\Omega(\omega^4)$ for this class. Hence, it seems that an improved bound for segments has to take advantage of straightness, usually this is a very hard task. However, the lower bound construction of [PT18] is using curves with multiple intersections. If we restrict to x-monotone pseudosegments, i.e. curves which intersect at most once, we may be able to improve the bound. At this point it would be an exaggeration to claim that we have an idea of how to prove a bound in this setting. But we have some experience in working with pseudolines and pseudosegments and look forward to fight with this problem.

Burling graphs are fascinating and we still feel that we do not have a full understanding of their structural properties. One concrete example that we have in mind is the presence of induced subdivisions of K_5. Let us bring some context. Scott [Sco97] conjectured in 1997 that for every graph H, the class defined by excluding all subdivisions of H as induced subgraphs is χ-bounded. We disproved this conjecture [PKK+14] when we showed that Burling graphs are segment intersection graphs. Indeed, they are triangle-free, they have arbitrarily large chromatic number and since they are segment intersection graphs they do not contain an induced subdivision of a 1-subdivided K_5 (or of any other 1-subdivided non-planar graph). Thus, Scott's conjecture is false for a 1-subdivision of any non-planar graph. Very recently Chalopin et al. (see [CELdM16]) were trying to understand for which graphs the Scott conjecture is true. It still remains open if it is true for K_5. We believe that Burling graphs do not contain an induced K_5 nor any subdivision of K_5. This would imply that the class of all graphs which contain no induced subdivision of K_5 is not chi-bounded.

II. Geometrically defined classes of graphs with linear structure.

Regarding the recognition complexity of stick graphs we have no clue whether we should expect that the problem is in **P** or is **NP**-complete. We will try in both directions. If a linear order of the vertices is prescribed, then it is easy to decide whether a corresponding stick representation exists. A valid linear order has to be a linear extension of the graph seen as a bipartite poset. To find an appropriate linear extension some kind of 'forcing relation' is needed. In [CFHW18] we have shown that induced cycles have a unique stick representation up to the choice of an extreme edge. We will study the connections of extreme edges for interfering cycles. With the help of computers we will identify other small graphs which have an essentially unique stick representation or no stick representation at all. Such structures can help to find appropriate forcing relations or to build gadgets for an **NP**-completeness proof. Another approach to the problem is to start with a result of Hell, Mohar, and Rafiey [HMR14] which implies that the superclass of bipartite hook graphs is polynomial time recognizable, just because the class is characterized by a forbidden pattern of length four (see Figure 1). We will try to strengthen the algorithm from [HMR14] as to avoid the use of hooks which have neighbors on both sides in the resulting representation.

Hell, Mohar, and Rafiey [HMR14] have shown that all graph classes defined by the existence of a vertex ordering avoiding a set of patterns of length three are recognizable in polynomial time. The situation for patterns of length four is more involved. Faber [Fab83] has shown that strongly chordal graphs are polynomially recognizable and that they are characterized by a set of patterns of length four. On the other hand graphs with $\chi \leqslant 3$ are characterized by a a set of patterns of length four and their recognition is **NP**-complete. We will investigate whether for some classes it is helpful that the subclass of bipartite graphs can be recognized efficiently. Special focus will be put on hook-graphs and grounded L-intersection graphs (cf. [JT18] for this class). We will use computers to identify small graphs which have an essentially unique vertex ordering avoiding the forbidden patterns and to identify some minimal graphs which are forbidden as induced subgraphs in the class. As in the case of stick graphs such structures can help in the design of recognition algorithms as well as in the construction of gadgets for an **NP**-completeness proof. An intriguing question in the area is due to Duffus et al. [DGR95], they conjecture that every class defined by a single 2-connected pattern (other than a complete graph) yields an **NP**-complete recognition problem. On four vertices there are only 9 instances. Maybe one of them can serve as a counterexample to the conjecture.

The conjecture of Heath, Leighton and Rosenberg that the queue number of planar graphs is constant is a real challenge. The conjecture has been settled in the positive for several subfamilies of planar graphs. Recent remarkable progress is due to Bekos, Ueckerdt et al. [BFG⁺18], they show that planar graphs of bounded degree have constant queue number. We have no real plan for attacking the general problem but we will try to combine ideas from [BFG⁺18] with techniques related to vertex orderings under constrains. This research will be carried out in close cooperation with Torsten Ueckerdt.

Stack layouts for posets have not been studied intensively. Sysło [Sys90] relates the stack-number with two other parameters, the jump- and the bump-number. It would be a nice master's thesis to study (experimentally) the performance of a local optimization algorithm which walks through linear extensions by performing adjacent flips. To attack the problem for planar posets we will first look at planar posets with 0 and 1, i.e., planar lattices. Structural insights from this study will guide further research.

Regarding the queue number of planar posets there are recent results by Knauer, Micek and Ueckerdt [KMU18]. Building on their examples we expect to be able to prove the conjecture of Heath and Pemmaraju for planar posets and possibly to disprove it for general posets.

III. Geometry and encodings.

We believe that we are in a privileged position to attack the old question of Nešetřil and Pudlák: Is the Boolean dimension of the planar posets bounded? Only recently Felsner, Micek and Mészáros [FMM17] proved that posets with cover graphs of bounded treewidth have bounded Boolean dimension. There are several new techniques buried in the proof of that result. They mainly make use of tree-like structures. The key to the planar case will be to develop similar techniques for grid-like structures. Another tool we keep in mind is the so-called *unfolding* of a poset (introduced by Streib and Trotter [ST14]). On the level of intuitions it works as follows: if a poset has large dimension, then it has a 'local' subposet which still has large dimension. This type of statement has a very simple and descriptive analogue in the world of graphs: if a graph G is connected and $\chi(G) > 2k$, then at least one distance level L (considered from any fixed vertex v) satisfies $\chi(G[L]) > k$. The 'locality' stems from the fact that in many cases (e.g. in minor-closed classes of graphs and specifically in the planar case) we can handle all the previous distance levels as if they were a single vertex.

We actually believe in the negative resolution of the question. Our strategy while working on this problem will be to push forward towards a positive resolution while observing the types of structures which remain unresolved. This may lead to a construction witnessing the negative resolution.

The question about Boolean dimension of the Boolean lattice of order n seems to be an innocent one. We have shared this problem over last two years with our colleagues all over the world. The exciting part is that the answer could be exactly n, which is a trivial upper bound. So far the best bound we (Felsner, Micek) have from the lower bound side is $\Omega(n/\log n)$. We do it with a short but relatively tricky counting argument. We are planning to push this idea up to its limits. Possibly, we will run some computational experiments testing the conjecture for small values of n.

4. References

A list of all publications cited in sections III.1 – III.3 of the Joint Project Description.

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