

A GRAPH-GRABBING GAME

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ABSTRACT. Two players share a connected graph with non-negative weights on the vertices. They alternately take the vertices (one in each turn) and collect their weights. The rule they have to obey is that the remaining part of the graph must be connected after each move. We conjecture that the first player can get at least half of the weight of any tree with an even number of vertices. We provide a strategy for the first player to get at least $1/4$ of an even tree. Moreover, we confirm the conjecture for subdivided stars. The parity condition is necessary: Alice gets nothing on a three-vertex path with all the weight at the middle. We suspect a kind of general parity phenomenon, namely, that the first player can gather a substantial portion of the weight of any “simple enough” graph with an even number of vertices.

1. INTRODUCTION AND CONJECTURE

A graph-grabbing game is played on a finite connected graph with non-negative weights on the vertices (from now on, simply a graph). There are two players: Alice and Bob. Starting with Alice, they take the vertices alternately one by one and collect their weights. The vertices taken are removed from the graph. The choice of a vertex to be played in each move is restricted by the rule that after each move the remaining vertices form a connected subgraph. In particular, playing on a tree, the players pick and remove leaves. Both players aim at maximizing their outcomes at the end of the game, when all vertices have been taken.

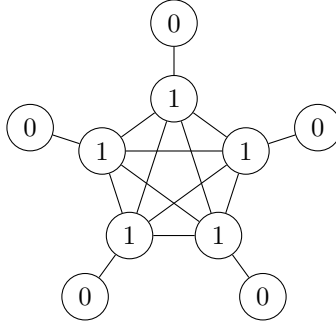
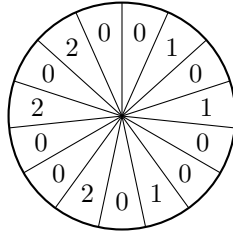
The first problem in Winkler’s puzzle book [7] is to prove that Alice can guarantee herself at least half of the weight of any path on an even number of vertices. The strategy is very simple: two-color the vertices and gather the heavier color. On the other hand, Alice cannot expect any positive score for odd paths: just consider a path on three vertices with all the weight at the middle. We believe this dichotomy of parities holds for all trees.

Conjecture 1. *Alice can secure at least $\frac{1}{2}$ of the weight of any tree with an even number of vertices.*

Further on we present a strategy for Alice to collect $\frac{1}{4}$ of the weight of any tree with an even number of vertices (Theorem 3). We also confirm the conjecture for subdivided stars, that is, trees with at most one vertex of degree greater than 2 (Theorem 4). Note that any odd tree on at least three vertices can be weighted so that Alice gets nothing (it suffices to put the whole weight at one non-leaf vertex).

This parity phenomenon does not hold for all graphs in the graph-grabbing game. There is no positive lower bound for the fraction of total weight that Alice can guarantee herself on any graph with an even number of vertices. Example 2 describes a sequence of even graphs on which Alice’s guaranteed gain tends to zero. Nevertheless, all such sequences known by the authors contain arbitrarily large cliques. There might exist a function $f(n) > 0$ such that Alice can secure at least $f(n)$ of the weight of any graph with an even number of vertices and with clique-size at most n .

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FIGURE 1. \mathbf{G}_5 : Alice cannot secure more than 1.FIGURE 2. Alice can guarantee herself at most $\frac{4}{9}$ of the pizza.

Example 2. $\mathbf{G}_n = (V, E, w)$ is a weighted graph with $2n$ vertices $V = \{a_1, \dots, a_n, b_1, \dots, b_n\}$. The b_i 's form a clique: $b_i E b_j$ for all $i \neq j$. The only neighbor of each a_i is b_i . The weights are distributed on the b_i 's: $w(a_i) = 0$ and $w(b_i) = 1$, and thus the total weight is n . Alice has no strategy to gather more than 1 from \mathbf{G}_n .

The graph-grabbing game on a cycle has been studied as the so-called pizza game: vertices are seen as slices of a pizza. The two-coloring argument shows that Alice can get at least half of an even pizza, and there are examples of odd pizzas where she can get only $\frac{4}{9}$ (see Figure 2). Winkler [6] conjectured in 2008 that Alice can secure at least $\frac{4}{9}$ of any pizza, and this has been proved by two independent groups of researchers [1, 3].

In the pizza game not only the remaining part but also the taken part is connected throughout the game. This leads to another generalization for graphs, in which the taken part is required to be connected. In this variant Alice starts by taking any vertex and then the players alternately pick vertices adjacent to already taken ones. The parity of the number of vertices makes a difference in this game as well, but this time odd graphs are better for Alice. In particular, Alice has a strategy to get at least $\frac{1}{4}$ of any odd tree, while even trees can be arbitrarily bad for her [4].

Independent research on generalizations of pizza game, also leading to the two aforementioned variants, has been carried out by Cibulka et al. [2]. They focus on connectivity and computational complexity issues. In particular, they provide a sequence of graphs of arbitrarily high connectivity and Alice's guaranteed gain tending to zero. They also prove that it is PSPACE-complete to decide who can get half of the weight of the graph in the graph-grabbing game. The complexity of the problem when input graphs are restricted to trees is unknown.

The name *graph-grabbing game* follows Moshe Rosenfeld, who proposed the game for trees and called it *gold-grabbing game* [5].

2. RESULTS

Theorem 3. *Alice can secure at least $\frac{1}{4}$ of the weight of any tree with an even number of vertices.*

Proof. A *two-colored tree* is a tree whose vertices are colored black or white so that no two adjacent vertices have the same color. We are going to prove that given a two-colored tree \mathbf{T} with an even number of vertices Alice can secure at least $\frac{1}{2}$ of the total weight of a color she chooses. This yields at least $\frac{1}{4}$ of the total weight of \mathbf{T} for Alice.

The proof goes by induction on the number of vertices of \mathbf{T} (only even numbers). For a tree with two vertices the statement is trivial. To get through the induction step we construct a strategy for Alice which leads the game to a point (after some Bob's move) at which the weight of black vertices she has taken is at least the weight of black vertices taken by Bob. We distinguish two cases.

Case 1: \mathbf{T} has a black leaf.

This is an easy case. Alice takes the heaviest available black vertex. With this move she does not uncover any new black vertex, so Bob can only respond with a white vertex or a non-heavier black vertex. In both scenarios Alice gains no less than Bob from the black part, and for the rest of the tree the induction hypothesis can be applied.

Case 2: All leaves of \mathbf{T} are white.

First observe that there must be a black vertex of degree greater than 2: if all black vertices have degree 2 then the total number of edges is even, so \mathbf{T} has an odd number of vertices, which contradicts the assumption. Let \mathbf{K} be the subtree of \mathbf{T} spanned by all black vertices of degree greater than 2. We call \mathbf{K} the *core* of \mathbf{T} , and we call connected components of $\mathbf{T} - \mathbf{K}$ simply *components*. The *root* of a component is its only vertex adjacent to the core. A similar argument with counting edges shows that a component with a white root has an odd number of vertices and a component with a black root has an even number of vertices (see Figure 3). Since all leaves of the core are black and have degree greater than 2, at least two of their (white) neighbors must be roots of components. Therefore, taking a vertex from the core is possible only after at least two components with an odd number of vertices have been entirely shared out.

Let $\mathbf{C}_1, \dots, \mathbf{C}_k$ be the components, and assume \mathbf{C}_1 is a component with an odd number of vertices and the least total weight of black vertices among all components with an odd number of vertices.

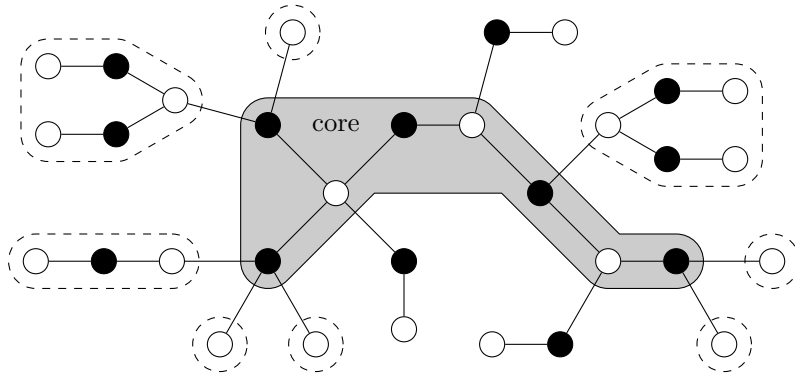


FIGURE 3. Illustration for Case 2. Odd components (marked with dashed lines) have white roots, even components have black roots.

Alice starts with any vertex from \mathbf{C}_1 . The following invariants are to be kept till the moment the induction hypothesis is applied:

- (i) For $i \neq 1$, all vertices in \mathbf{C}_i available for Bob are white.
- (ii) No component with an odd number of vertices other than \mathbf{C}_1 has been entirely shared out.

Subsequent moves of Alice depend on what Bob has just played. Observe that by (ii) no vertex from the core is available. Thus, Bob can choose only from the vertices in the components.

Case 1: Bob takes a vertex from \mathbf{C}_1 .

Alice takes another vertex of \mathbf{C}_1 . Such a vertex exists as \mathbf{C}_1 has an odd number of vertices. Clearly, both invariants are preserved.

Case 2: Bob takes a vertex v from \mathbf{C}_i with $i \neq 1$.

If v is not the root of \mathbf{C}_i then Bob's move uncovers exactly one black vertex in \mathbf{C}_i which is now a new leaf. Alice takes it. It may be the root of \mathbf{C}_i but then \mathbf{C}_i has an even number of vertices, so taking the last vertex of \mathbf{C}_i does not violate (ii). All vertices that remain available in \mathbf{C}_i are white exactly as (i) states.

If v is the root of \mathbf{C}_i then \mathbf{C}_i has an odd number of vertices and all of them have been taken. Note that all black vertices taken by Bob are in \mathbf{C}_1 , and Alice has taken all black vertices from \mathbf{C}_i . Since the weight of black vertices in \mathbf{C}_i is at least the weight of black vertices in \mathbf{C}_1 , Alice has collected at least as much as Bob from the black part. For the remaining tree the induction hypothesis is applied. \square

Theorem 4. *Alice can secure at least $\frac{1}{2}$ of the weight of any subdivided star with an even number of vertices.*

Proof. Let \mathbf{T} be the given subdivided star and w be the weight function. The proof goes by induction on the number of vertices of \mathbf{T} , just like the proof of Theorem 3. If \mathbf{T} is a path then the statement is easy, as discussed in the Introduction. Thus suppose \mathbf{T} is not a path.

Let v be the vertex of \mathbf{T} of degree $d \geq 3$. Other vertices of \mathbf{T} form d paths $u_{k1}u_{k2}\dots u_{k\ell_k}$ for $k = 1, \dots, d$, which we call *arms* of \mathbf{T} . Here u_{11}, \dots, u_{d1} are the d leaves of \mathbf{T} and $u_{1\ell_1}, \dots, u_{d\ell_d}$ are the d neighbors of v .

We define the *alternating weight* of any prefix $u_{k1}\dots u_{kj}$ of the k th arm to be

$$\hat{w}(u_{k1}\dots u_{kj}) = \sum_{i=1}^j (-1)^i w(u_{ki}).$$

The idea behind this definition is the following. Suppose at some point of the game the players start to play on the k th arm and they alternately take the vertices u_{k1}, \dots, u_{kj} . Then $\hat{w}(u_{k1}\dots u_{kj})$ measures the relative gain of the second player from this line of play.

We say that $u_{k1}\dots u_{kj}$ has the *even prefix property* if for any even $i \leq j$ we have $\hat{w}(u_{k1}\dots u_{ki}) > 0$. The k th arm is called *bad* if it has the even prefix property and its length ℓ_k is even. Otherwise the k th arm is *good*. The intuition is that it is (almost) never profitable to start playing in a bad arm, because the other player can always continue in the same arm and gain more.

We will present two complementary strategies: one for the case of at most two good arms, and one for the other case. Only the latter uses induction.

Strategy 1: \mathbf{T} has at most two good arms.

Since each bad arm has even length and there are even number of vertices in total, exactly one arm has odd length. Assume without loss of generality that arm 1 has odd length and that arms $3, \dots, d$ are bad. Two-color the vertices of \mathbf{T} properly with black or white. One of the colors, say, black carries at least half of the total

weight of \mathbf{T} . We will show that Alice can secure at least the total weight of all black vertices.

Case 1: u_{11} is black and u_{21}, \dots, u_{d1}, v are white.

In this case the strategy for Alice is just to take the only black leaf available. This leads to a position where all leaves are white. Bob is forced to take a white leaf and thus to uncover a new black leaf for Alice. This way Alice gathers all black vertices.

Case 2: u_{11} is white and u_{21}, \dots, u_{d1}, v are black.

Initially arms 1, 2 are marked *active* and arms 3, \dots , d are marked *passive*. Alice starts by taking u_{21} . After Bob takes u_{ki} , Alice responds in the same arm by taking $u_{k,i+1}$ unless $i = \ell_k$. In the latter case Alice chooses any remaining passive arm, marks it active and takes its available vertex. If there are no more passive arms, the remaining part of \mathbf{T} is a path and therefore Alice can easily collect all remaining black vertices. This strategy guarantees that Alice takes black and Bob takes white vertices, except that playing in arms 3, \dots , n when passive Alice takes white vertices and Bob takes black ones. Vertices played when an arm is passive form an even prefix of it. Since arms 3, \dots , n are bad, the weight of white is greater than the weight of black in these prefixes. Therefore, Alice scores at least the total weight of all black vertices.

Strategy 2: \mathbf{T} has at least three good arms.

The strategy we are going to present now leads the game to a point at which Alice has gathered at least as much as Bob after some Bob's move. Then we apply induction hypothesis to show that Alice can secure at least half of the remaining part of \mathbf{T} .

For any good arm k let r_k be such that

- * r_k is odd;
- * $u_{k1} \dots u_{kr_k}$ has the even prefix property;
- * $\hat{w}(u_{k1} \dots u_{kr_k})$ is maximized.

Assume without loss of generality that arm 1 minimizes $\hat{w}(u_{k1} \dots u_{kr_k})$ among all good arms k . Alice starts by taking u_{11} . Then, each time after Bob has taken a vertex u_{ki} , Alice responds in the same arm by taking $u_{k,i+1}$ until

- (a) Bob has just taken u_{1i_1} such that $\hat{w}(u_{11} \dots u_{1i_1}) \leq 0$, or
- (b) Bob has just taken u_{kr_k} from a good arm $k \neq 1$.

We will prove that

- * when (a) or (b) happens, Alice has just gathered at least as much as Bob and therefore can play inductively in the remaining part of \mathbf{T} ;
- * if Bob takes u_{ki} and neither of (a), (b) happens then $i < \ell_k$ so the vertex $u_{k,i+1}$ exists;
- * Bob cannot take v .

Suppose that (a) happens. We may assume that vertices were taken from arms 1 through c for some $1 \leq c \leq d$. Let $u_{2i_2}, \dots, u_{ci_c}$ be the last vertices taken (by Alice) from arms 2, \dots , c respectively. For good arms k we have $i_k < r_k$, as otherwise (b) would happen before. Therefore, by the even prefix property in the definitions of r_k and of a bad arm, $\hat{w}(u_{k1} \dots u_{ki_k}) > 0$ for every $k = 2, \dots, c$. Observe that $\hat{w}(u_{k1} \dots u_{ki_k})$ is the difference between Alice's and Bob's gain in arm k , and the difference between Alice's and Bob's gain in arm 1 equals $-\hat{w}(u_{11} \dots u_{1i_1}) \geq 0$.

Now, suppose that (b) happens. The same argument as above shows that Alice has scored more than Bob in arms other than 1 and k . Let i_1 be the last vertex taken (by Alice) from arm 1. Since (a) did not happen before, $u_{11} \dots u_{1i_1}$ has the

even prefix property. Therefore,

$$\hat{w}(u_{11} \dots u_{1i_1}) \leq \hat{w}(u_{11} \dots u_{1r_1}) \leq \hat{w}(u_{k1} \dots u_{kr_k}).$$

It suffices to observe that the difference between Alice's and Bob's gain in arms 1 and k together is $\hat{w}(u_{k1} \dots u_{kr_k}) - \hat{w}(u_{11} \dots u_{1i_1}) \geq 0$.

Suppose that Bob takes $u_{k\ell_k}$ and neither of (a), (b) happens. If $k = 1$ then ℓ_1 is even and $u_{11} \dots u_{1\ell_1}$ has the even prefix property, as otherwise (a) would happen now or before. This contradicts the fact that arm 1 is good. If $k \neq 1$ then ℓ_k is odd, so arm k is good. Therefore, (b) would happen now or before.

Finally, to show that Bob cannot take v , observe that taking v is possible only if all arms except one are entirely taken out. However, there are at least three good arms, of which only arm 1 can be entirely taken out before (b) happens. \square

REFERENCES

- [1] Josef Cibulka, Jan Kynčl, Viola Mészáros, Rudolf Stolař, and Pavel Valtr. Solution to Peter Winkler's pizza problem. In Jiří Fiala, Jan Kratochvíl, and Mirka Miller, editors, *Combinatorial Algorithms*, volume 5874 of *Lecture Notes in Computer Science*, pages 356–367. Springer, 2009.
- [2] Josef Cibulka, Jan Kynl, Viola Mszros, Rudolf Stola, and Pavel Valtr. Graph sharing games: Complexity and connectivity. In Jan Kratochvíl, Angsheng Li, Jir Fiala, and Petr Kolman, editors, *Theory and Applications of Models of Computation*, volume 6108 of *Lecture Notes in Computer Science*, pages 340–349. 2010.
- [3] Kolja Knauer, Piotr Micek, and Torsten Ueckerdt. How to eat $\frac{4}{9}$ of a pizza. to appear in *Discrete Mathematics*.
- [4] Piotr Micek and Bartosz Walczak. Parity in graph sharing games. submitted.
- [5] Moshe Rosenfeld. A gold-grabbing game. Open Problem Garden, http://garden.irmacs.sfu.ca/?q=op/a_gold_grabbing_game.
- [6] Peter M. Winkler. problem posed at Building Bridges, a conference in honour of 60th birthday of László Lovász, Budapest, 2008.
- [7] Peter M. Winkler. *Mathematical puzzles: a connoisseur's collection*. A K Peters Ltd., Wellesley, MA, 2004.

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