ON-LINE CHAIN PARTITIONING OF UP-GROWING INTERVAL ORDERS

PATRICK BAIER, BARTŁOMIEJ BOSEK, AND PIOTR MICEK

Abstract. On-line chain partitioning problem of on-line posets has been open for the past 20 years. The best known on-line algorithm uses $\frac{5^w-1}{4}$ chains to cover poset of width w. Felsner [3] introduced a variant of this problem considering only up-growing posets, i.e. on-line posets in which each new point is maximal at the moment of its arrival. He presented an algorithm using $\binom{w+1}{2}$ chains for width w posets and proved that his solution is optimal. In this paper, we study on-line chain partitioning of up-growing interval orders. We prove lower bound and upper bound to be $2w-1$ for width w posets.

1. Introduction

An on-line chain partitioning algorithm receives as input a partially ordered set (poset) in an on-line way. This means that the elements of the poset are taken one by one from some externally determined list. Being presented with a new element the algorithm learns the comparability status of previously presented elements to the new one. Based on this knowledge the algorithm assigns the new element to a chain in an irrevocable manner. The performance of an on-line chain partitioning algorithm is measured by comparing the number of chains used with the number of chains needed by an optimal off-line algorithm, i.e. with the width of the poset (see Theorem 1). For the terminology of partially ordered sets as well as for the proof of Theorem 1 we refer the reader to [7].

Theorem 1 (Dilworth [1])**.** *If $\mathcal{P} = (P, \leqslant)$ is a poset and width$(P) = w$, then there exists a covering $P = \alpha_1 \cup \ldots \cup \alpha_w$, where α_i's are chains.*

All on-line problems considered in this paper can be viewed as two-person games. We call the players Algorithm and Spoiler. Algorithm represents an on-line algorithm and Spoiler represents an adaptive adversary. The game is played in rounds. During each round Spoiler

Date: January 20, 2007.

Key words and phrases. on-line, chain partitioning, interval order.

The authors are scholars of the project which is co-financed from the European Social Fund and national budget in the frame of The Integrated Regional Operational Programme.

introduces a new point x to the poset and describes comparabilities between x and the points from all previous rounds. Algorithm responds by assigning x to a chain. The most important feature of on-line games is that Algorithm's previous moves (decisions) restrict his actual possibilities.

The *value of the on-line chain partitioning problem* for width w posets is the largest integer $CP(w)$ so that there is a strategy of presenting points that forces any algorithm to use $CP(w)$ chains. Note that we may as well define $CP(w)$ as the least integer so that there is an on-line algorithm that never uses more chains. An unpublished argument of Szemerédi proves a lower bound and Kierstead [5] showed an upper bound. Together, this means, we know

$$\binom{w+1}{2} \leqslant CP(w) \leqslant \frac{5^w - 1}{4}.$$

This is already a complicated result, and no progress has been made on this problem for the last 20 years. Felsner [3] introduced a variant of the chain partitioning problem. He restricts possible inputs by the rule that the sequence in which elements are released is a linear extension of the poset, i.e. a comparability of a new element x to the former y's has to be of the form $y < x$. In other words, each new point is maximal at the moment of its arrival. On-line posets with this property are called up growing posets and the value of the chain partitioning problem of up-growing, width w posets, defined similarly to the previous one, is denoted by $CPU(w)$. Felsner [3] proved that

$$CPU(w) = \binom{w+1}{2}.$$

In this paper we address the on-line chain partitioning problem for a special, but important class of posets that can be represented on a real line by intervals.

Definition 2. *A poset $\mathcal{P} = (P, \leqslant)$ is an interval order if there is a function I assigning to each element $x \in P$ a closed interval $I(x) = [l_x, r_x]$ of the real line $\mathbb{R}$ so that for all $x, y \in P$ we have $x < y$ in P iff $r_x < l_y$ in $\mathbb{R}$. A family of all those intervals is called an interval representation of $\mathcal{P}$.*

From the paper of Kierstead and Trotter [6] one easily infers that the value $CPI(w)$ of on-line chain partitioning problem of interval orders is given by

$$CPI(w) = 3w - 2.$$

This paper is devoted to determine the analogous value of the on-line chain partitioning of up-growing interval orders denoted by $CPUI(w)$. We prove that

$$CPUI(w) = 2w - 1.$$

Throughout this paper, we say that distinct points x, y are *comparable* if either $x < y$ or $y < x$. Otherwise, we say x and y are *incomparable* and write $x \parallel y$. If $\mathcal{P} = (P, \leqslant)$ and $X \subseteq P$ then by *closed/open upset* of X in P we mean

$$\begin{aligned} X{\Uparrow}_{\mathcal{P}} &:= \{p \in P : \text{ there is } x \in X \text{ so that } x \leqslant p\}\,, \\ X{\uparrow}_{\mathcal{P}} &:= \{p \in P : \text{ there is } x \in X \text{ so that } x < p\}\,. \end{aligned}$$

Dually we define downsets $X{\Downarrow}_{\mathcal{P}}$ and $X{\downarrow}_{\mathcal{P}}$ of X in P. If $X = \{x\}$, we prefer to write $x{\uparrow}_{\mathcal{P}}$ instead of $\{x\}{\uparrow}_{\mathcal{P}}$. The reference to the poset $\mathcal{P}$ is often omitted when $\mathcal{P}$ is clear from the context.

Instead of referring directly to the definition of an interval order there are settings in which we use the following characterization theorem, a proof of which can be found in [7].

Theorem 3 (Fishburn [4])**.** *Let $\mathcal{P} = (P, \leqslant)$ be a poset. Then the following statements are equivalent:*

(1) *$\mathcal{P}$ is an interval order.*
(2) *$\mathcal{P}$ is a $(2 + 2)$-free poset, i.e., P does not contain elements $a, b, c, d \in P$ such that: $a < b$, $c < d$, $a \parallel d$ and $c \parallel b$.*
(3) *for $x, y \in P$ we have $x{\uparrow} \subseteq y{\uparrow}$ or $y{\uparrow} \subseteq x{\uparrow}$.*
(4) *for $x, y \in P$ we have $x{\downarrow} \subseteq y{\downarrow}$ or $y{\downarrow} \subseteq x{\downarrow}$.*

2. Lower bound

Theorem 4. *There is no on-line algorithm for chain partitioning of up-growing interval orders using less than $2w - 1$ chains to cover posets of width w, i.e., $2w - 1 \leqslant CPUI(w)$.*

To prove Theorem 4 one should provide a strategy for Spoiler building a poset of width w and forcing Algorithm to use at least $2w - 1$ chains. We consider points presented by Spoiler as intervals (representing these points), but before introducing a new interval Spoiler may change the order of right endpoints of intervals representing points with the same open upset, i.e. the order of right endpoints lying between the same two left endpoints or lying to the right of the rightmost left endpoint. This is allowed because Spoiler is not obliged to present a representation of the interval order and such modified family of intervals still represents the same interval order.

In following we abuse standard notation using terminology of partially ordered sets for interval representations. In particular, considering interval representation I of an interval order we say that $I(x)$ is above (to the right of) $I(y)$ if $x < y$.

Claim. *Let M be an antichain of size w and let $2 \leqslant v \leqslant w$. There is a strategy $S(v, M)$ for Spoiler to build in an up-growing way an extension Q of M such that*

I1: *Q has width v and has v maximal intervals. $M \cup Q$ has width w.*

I2: *Algorithm is forced to use at least $2v - 2$ chains in Q, among them at least $v - 1$ chains not used in M.*

I3: *There is a minimal interval q in Q such that $q > m$ for all $m \in M$.*

I4: *All intervals in M that are not covered by chains used in Q have the same right endpoint r while the other intervals in M have right endpoints to the left of r.*

Proof. We use induction on v to construct recursively a strategy $S(v, M)$ for Spoiler satisfying all four conditions I1-I4. First, we provide a strategy $S(2, M)$ for an arbitrary antichain M of width w:

Without loss of generality let M be covered by $1, \dots, w$. First, Spoiler makes equal all right endpoints of intervals from M. Now, Spoiler puts a new interval x to the right of all the intervals of M. Algorithm will either decide to use one of the chains from M, w.l.o.g. chain 1. Then Spoiler decreases the right endpoint of an interval in M covered by 1, say m, and introduces a new interval y only above m (see the left part of Figure 1). Or, Algorithm already covers x by a new chain. In this case Spoiler puts y above each interval of M (see the right part of Figure 1). No matter which chain Algorithm uses, the invariants are satisfied.

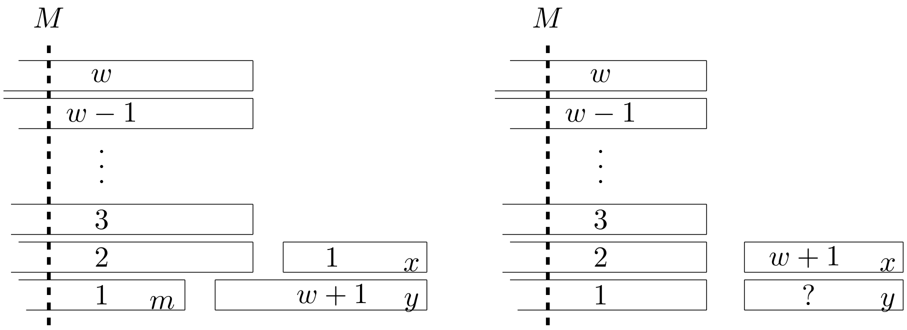

FIGURE 1. $S(2, M)$

The induction step proceeds from v to $v + 1$ for $v < w$ and an arbitrary antichain M of width w. First, Spoiler extends M by Q' calling recursively the procedure $S(v, M)$. Thus, Algorithm uses at least $2v - 2$ chains to cover Q' and at least $v - 1$ among them are not used in M. Let A be the antichain of all maximal points in Q'. By I1 for Q' we have $|A| = v$. Now, Spoiler runs $S(v, A)$ producing an extension Q'' of A. According to the induction hypothesis Algorithm is forced to use in Q'' at least $v - 1$ chains not used in A. All these together yield that at least $(v - 1) + v = 2v - 1$ chains are used in $Q' \cup Q''$ and among them at least $v - 1$ chains are not used in M. Let

N be the subset of intervals of M whose chains are used in $Q' \cup Q''$; M_1 be the subset of M whose chains are used in Q' and $M_2 := M - M_1$. Note that $M_1 \subseteq N$. Now, there are two cases:

Case 1. Algorithm used at least v chains from M in $Q' \cup Q''$, i.e. $|N| \geqslant v$. In this case Spoiler introduces an interval x above all intervals in N and incomparable to the other intervals. To do that Spoiler first

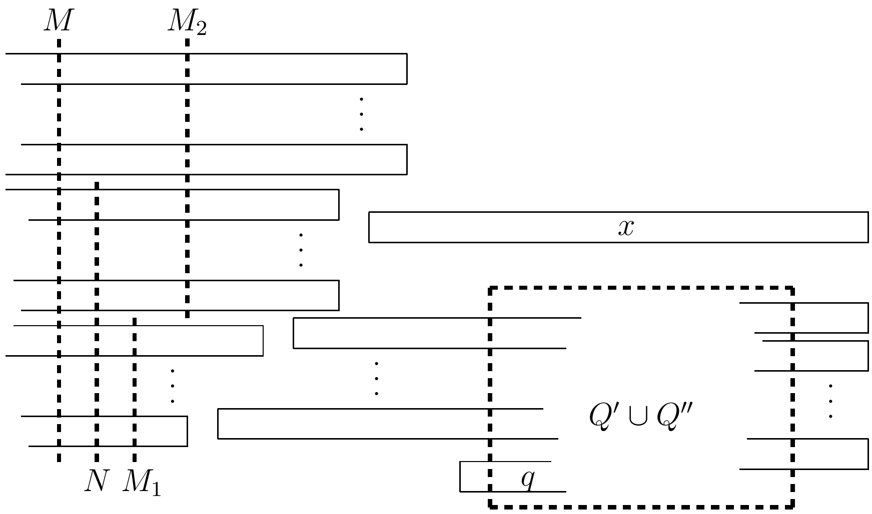

FIGURE 2. Case 1: $S(v+1, M)$

rearranges the right endpoints of intervals in M so that each interval in N ends before any interval in $M - N$. To see that such rearrangement is possible, first note that by I4 for Q' intervals from M_1 are already enough to the left. Moreover, by I4, for Q' the right endpoints of intervals from M_2 coincide. Thus, Spoiler may shorten intervals in $M_2 \cap N$ to end them to the left of than these in $M_2 - N$.

Now, knowing that such an x can be introduced by Spoiler we put $Q := Q' \cup Q'' \cup \{x\}$ and we argue that M extended with Q satisfies conditions I1-I4.

ad I1: $M \cup Q'$ has width w, Q' has width v and v maximal points, forming the set A, which is the base for Q''. This together with the fact that $A \cup Q''$ has width v implies that $\mathit{width}(M \cup Q' \cup Q'') = w$, $\mathit{width}(Q' \cup Q'') = v$ and therefore $\mathit{width}(Q' \cup Q'' \cup \{x\}) = v+1$. Obviously, $Q' \cup Q'' \cup \{x\}$ has $v+1$ maximal points: all v maximal points of Q'' and the point x. It remains to prove that $\mathit{width}(M \cup Q' \cup Q'' \cup \{x\}) = w$. The only region where x could increase the width to become greater than w is where $x \parallel M - N$. But since q is above $M - N$ the size of such an antichain is bounded by $|M - N| + 1 + (v-1) \leqslant (w - v) + 1 + (v - 1) = w$.

ad I2: Algorithm has to cover x by a completely new chain, i.e. not used in $M \cup Q' \cup Q''$. Thus, there are at least $v + (v-1) + 1 =$

$2(v+1)-2$ chains used in Q and $(v-1)+1 = (v+1)-1$ chains in Q and not used in M.

ad I3: The point witnessing I3 for Q' witnesses I3 also for Q.

ad I4: The rearrangement of right endpoints of intervals in M made by Spoiler before introducing x guarantees that I4 still holds.

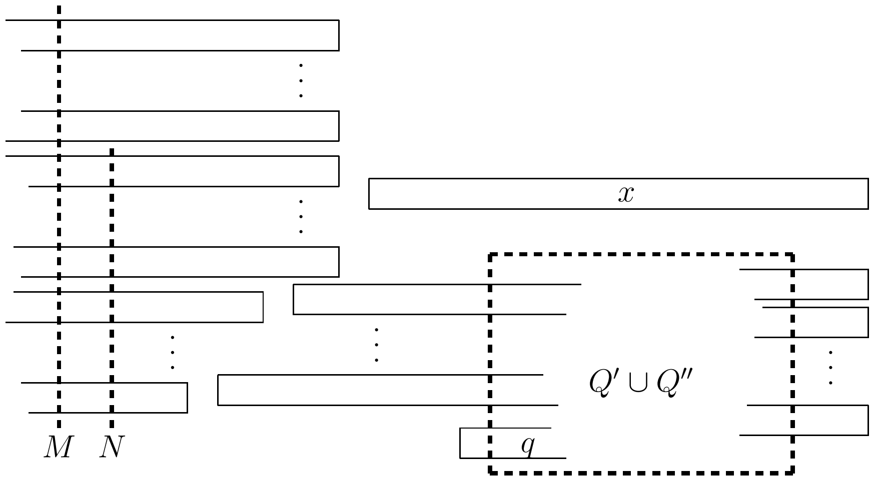

FIGURE 3. Case 2: $S(v+1, M)$

Case 2. Algorithm used at most $v-1$ chains from M to cover $Q' \cup Q''$. This means that at least v chains not used in M are used in $Q' \cup Q''$. In this case, Spoiler puts new point x above all intervals in M and only above them. Put $Q := Q' \cup Q'' \cup \{x\}$. Invariants I1 and I3 may proved in a very similar way as in Case 1. To prove I2, observe that while covering x Algorithm may use a completely new chain or a chain used in M and not used in Q. In both cases we get $2v-1+1 = 2(v+1)-2$ chains in $Q' \cup Q'' \cup \{x\}$ and already knowing that there are at least v chains used in $Q' \cup Q''$ not from M, we are done. The condition I4 is trivially kept. □

Now, to prove Theorem 4, Spoiler starts with an antichain M with w points. Next it uses strategy $S(w, M)$ to build an extension Q of M and force Algorithm to use at least $w-1$ chains not used in M. This shows that at least $2w-1$ chains have to be used, so that $CPUI(w) \geqslant 2w-1$.

3. Upper bound

Before presenting an algorithm that proves our upper bound to be $2w-1$ we introduce some definitions.

By a *maximum antichain* in $\mathcal{P}$ we mean an antichain that is maximal with respect to the number of its elements, or in other words an antichain with exactly $width(P)$ elements. An antichain $A \subseteq P$ is *high* in $\mathcal{P}$ if for any antichain $B \subseteq A\Uparrow$ we have $A = B$ or $|B| < |A|$. Dilworth

[2] proved that in a finite poset $\mathcal{P}$ there is exactly one maximum and high antichain. This unique antichain is denoted by $HMA(P)$.

Theorem 5. *There is an on-line algorithm for chain partitioning of up-growing interval orders that uses at most $2w-1$ chains, where w is the width of the poset, i.e., $CPUI(w) \leqslant 2w-1$.*

Proof. Our algorithm maintains an auxiliary data structure S that depends on an up-growing interval order presented as an input $\mathcal{P} = (P, \leqslant)$ as well as on the already built covering of $\mathcal{P}$. When $\mathcal{P}$ expands to $\mathcal{P}^+ = (P \cup \{x\}, \leqslant)$ by a new, maximal point x then our algorithm modifies S to get a new structure S^+ for $\mathcal{P}^+$. The chain for x can be easily read from S^+. Put $S = (P, \leqslant, L_1, \ldots, L_w, \alpha_1, \ldots, \alpha_w, \beta_1, \ldots, \beta_w)$, where $w = width(P)$ and:

I1: $L_1, \ldots, L_w$ are high antichains in P such that $L_1{\Uparrow} \subseteq \ldots \subseteq L_w{\Uparrow}$ and $|L_i| = i$,

I2: $\alpha_1, \ldots, \alpha_w, \beta_1, \ldots, \beta_w$ forms a chain partition of P,

I3: $\alpha_i \subseteq L_i{\Downarrow} - L_{i-1}{\Uparrow}$, $\beta_i \subseteq L_i{\downarrow}$,

where for a further simplification we put $L_0 = \emptyset$.

For an on-line algorithm it is important that chains generated to cover P^+ expand those for P. This will be secured by the condition that $\alpha_i's$ and $\beta_i's$ grow in time as shown in lines (A7), (A8), (A20), (A21) of the algorithm we construct. Antichains $L_1, \ldots, L_w$ may be seen as levels of the poset P. Two consecutive levels L_{i-1}, L_i determine our chains α_i and β_i as described in I3.

Before describing our algorithm we note that all segments of the form $X{\Downarrow}, X{\downarrow}, X{\Uparrow}, X{\uparrow}$ are considered in $\mathcal{P}^+$. Whenever we need to refer to an upset in P we write $X{\Uparrow} \cap P$, while for $X \subseteq P$ downsets $X{\Downarrow}$ are the same in P and P^+.

Algorithm.

(A1) **if** $width(P) < width(P^+)$ **then Case A else Case B**

(A3) **Case A**
(A4) **for** $i := 1$ **to** w **do**
(A5) **begin**
(A6) $L_i^+ := L_i$
(A7) $\alpha_i^+ := \alpha_i$
(A8) $\beta_i^+ := \beta_i$
(A9) **end**
(A10) $L_{w+1}^+ := L_w \cup \{x\}$
(A11) $\alpha_{w+1}^+ := \{x\}$
(A12) $\beta_{w+1}^+ := \emptyset$

(A14) **Case B**

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(A15)  {If width(P) = width(P⁺) then x ∈ L_w↑.}
(A16)  i_0 := min{i ∈ N : x ∈ L_i↑}
(A17)  for i:=1 to w do
(A18)      begin
(A19)      L_i^+ := HMA(L_i⇑)
(A20)      if i ≠ i_0 then α_i^+ := α_i else α_i^+ := β_i ∪ {x}
(A21)      if i ≠ i_0 then β_i^+ := β_i else β_i^+ := α_i
(A22)      end
```

First of all our algorithm checks whether the new point x enlarges the width of the poset. If it does then the algorithm may use two new chains (see (A11),(A12)). We prove that our algorithm upgrading S to $S^+ = (P^+, \leqslant, L_1^+, \ldots, L_{w^+}^+, \alpha_1^+, \ldots, \alpha_{w^+}^+, \beta_1^+, \ldots, \beta_{w^+}^+)$ keeps the properties I1-I3 invariant. The proof of this splits into two parts corresponding to cases A and B, respectively.

Case A: In this setting, according to (A1), we have $\mathit{width}(P^+) > \mathit{width}(P)$. Since one point may increase the width of the poset at most by 1, we have

$$\mathit{width}\big(P^+\big) = \mathit{width}(P) + 1.$$

An additional new level L_{w+1}^+ is defined by (A10). The other levels are unchanged, see (A6). The new point x is covered by a new chain α_{w+1}^+, see (A11). Chain β_{w+1}^+ is defined as an empty set, see (A12).

By I1 we know that $L_w = \mathit{HMA}(P)$. Since point x increased the width, we know that x is incomparable with some maximum antichain A in P. Of course, we have $L_w = \mathit{HMA}(P) \subseteq A\Uparrow$. Thus, from $x \notin L_w\uparrow$ and from the fact that x is maximal in P^+(since presented poset must be up-growing) we get that $L_{w+1}^+ = L_w \cup \{x\}$ is a high antichain in P^+.

Obviously, $L_i^+\Uparrow \subseteq L_{i+1}^+\Uparrow$. Combining this and $x \notin L_w\uparrow$ we obtain $x \notin L_i\uparrow$ for all $i = 1, \ldots, w$. Thus,

$$L_i^+\Uparrow = L_i\Uparrow = L_i\Uparrow \cap P.$$

Since the L_i are high in P (see I1 for S) we immediately get that the L_i^+ are high in P^+.

The condition describing the cardinalities of L_i^+'s as well as these concerning sets α_i^+ and β_i^+ trivially follow from those for S.

Case B: By (A1) we have $\mathit{width}(P^+) = \mathit{width}(P)$. Thus, there is a point in L_w comparable with x. Since x is maximal in P^+ we get that $x \in L_w\uparrow$. Now, we know that i_0 in line (A16) is well-defined as the set under $\mathit{min}()$ function is not empty.

Claim. *$L_1^+, \ldots, L_w^+$ are high antichains in P^+. Moreover, $L_1^+\Uparrow \subseteq \ldots \subseteq L_w^+\Uparrow$ and $\left|L_i^+\right| = i$.*

Proof. Since $L_i^+ = HMA(L_i\Uparrow)$ we get that the levels $L_1^+, \ldots, L_w^+$ of the poset P^+ are high antichains in P^+. To prove that $L_1^+\Uparrow \subseteq \ldots \subseteq L_w^+\Uparrow$ we need the following remark.

Remark. *For a high antichain H and a maximum antichain M in a poset $\mathcal{Q} = (Q, \leqslant)$ we have $H \subseteq M\Uparrow$.*

Proof. Since M is a maximum antichain in Q, we know that each $x \in Q$ is comparable to some $h \in M$. All we have to show is that no element from H lies strictly below M. Suppose to the contrary that the set $H_1 = M\downarrow \cap H$ is non-empty. Put

$$\begin{aligned} H_2 &= H - H_1, \\ M_1 &= H_1\uparrow \cap M, \\ M_2 &= M - M_1. \end{aligned}$$

Observe that, both $M_1 \cup H_2$ as well as $H_1 \cup M_2$ are antichains and

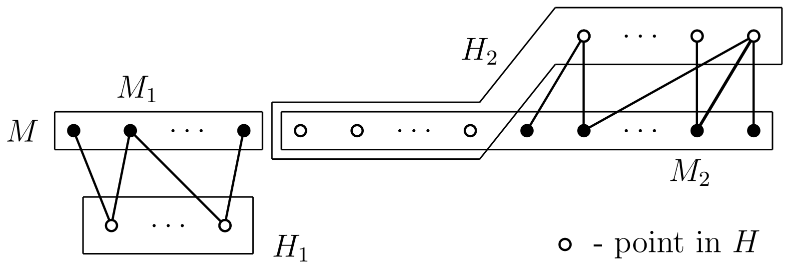

FIGURE 4

$M_1 \cup H_2 \subseteq H\Uparrow$. Now, if $|M_1| \geqslant |H_1|$ then $M_1 \cup H_2$ would have at least the same number of elements as H. This is impossible as H is high. Thus $|M_1| < |H_1|$. But then the antichain $H_1 \cup M_2$ has more elements than the maximum antichain M itself. This contradiction proves the remark. □

Now, we are ready to prove that $L_1^+\Uparrow \subseteq \ldots \subseteq L_w^+\Uparrow$. Observe that

$$\begin{aligned} L_i^+ &= HMA(L_i\Uparrow) && \text{by (A19)} \\ &\subseteq L_i\Uparrow \\ &\subseteq L_{i+1}\Uparrow\,. && \text{by I1 for } S \end{aligned}$$

By I1 we already know that L_i^+ is high in P^+. But $L_i^+ \subseteq L_{i+1}\Uparrow$ so that L_i^+ is high in $L_{i+1}\Uparrow$. Applying Remark with $Q = L_{i+1}\Uparrow$ we get that $L_i^+ \subseteq HMA(L_{i+1}\Uparrow)\Uparrow = L_{i+1}^+\Uparrow$.

To prove that $\left|L_i^+\right| = |L_i|$ (i.e. I1 for S^+) we first consider the width of $L_i\Uparrow$. By I1 for S we know that L_i is high in P. Thus for any antichain $A \subseteq L_i\Uparrow$ we have $A = L_i$ or $|A - \{x\}| < |L_i|$. This implies that $width(L_i\Uparrow) = |L_i|$, and consequently

$$\begin{aligned} \left|L_i^+\right| &= |HMA(L_i\Uparrow)| && \text{by (A19)} \\ &= width(L_i\Uparrow) \\ &= |L_i| \\ &= i. && \text{by I1 for } S \end{aligned} \tag{1}$$

□

Claim. $\alpha_1^+, \ldots, \alpha_w^+, \beta_1^+, \ldots, \beta_w^+$ *forms a chain partition of* P^+.

Proof. After updating L_i's to L_i^+'s algorithm defines the sets $\alpha_1^+, \ldots, \alpha_w^+, \beta_1^+, \ldots, \beta_w^+$. It turns out (see (A20),(A21)) that the only chains modified are at i_0:

$$\begin{aligned} \alpha_{i_0}^+ &= \beta_{i_0} \cup \{x\}, \\ \beta_{i_0}^+ &= \alpha_{i_0}. \end{aligned}$$

The Claim is obvious except the fact that $\alpha_{i_0}^+$ is a chain. Thus, we need to prove that

$$b < x, \qquad \text{for all } b \in \beta_{i_0}. \tag{2}$$

By I3 for S we have

$$\beta_{i_0} \subseteq L_{i_0}\downarrow . \tag{3}$$

Let $b \in \beta_{i_0}$. By (3) we obtain $l \in L_{i_0}$ such that $b < l$. On the other hand since $x \in L_{i_0}\uparrow$ (see (A16)) we get $l' \in L_{i_0}$ such that $l' < x$. If $b \parallel x$ then points b, l, l', x would form a $2+2$ configuration, which is forbidden in an interval order. This proves (2). □

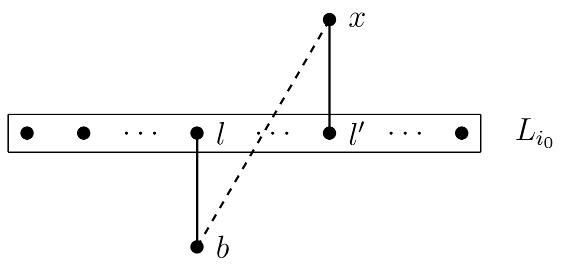

FIGURE 5

As we can see the idea of the algorithm is to ensure that the chain β_i is able to cover any point over the level L_i (we may say that β_i secures L_i). Thus, when a new point x arrives over L_i then chain β_i may be used to cover it. But Algorithm must choose such i that chain $\beta_i^+ = \alpha_i$ (see (A21)) will be able to secure the new level L_i^+. It will turn out that the choice of i_0 as in (A16) it is the good one.

Before proving I3 for S^+ we present correlations between open upsets of old (L_i's) and new (L_i^+'s) levels.

Claim.

$$\begin{array}{ccccccccccccccc}
L_1\Uparrow & \subseteq \ldots \subseteq & L_{i_0-1}\Uparrow & \subseteq & L_{i_0-1}\Uparrow \cup \{x\} & \subseteq & L_{i_0}\Uparrow & \subseteq & L_{i_0+1}\Uparrow & \subseteq \ldots \subseteq & L_w\Uparrow \\
\shortparallel & & \shortparallel & & \shortparallel & & & & \cup & & \cup \\
L_1^+\Uparrow & \subseteq \ldots \subseteq & L_{i_0-1}^+\Uparrow & \subseteq & L_{i_0}^+\Uparrow & & \subseteq & & L_{i_0+1}^+\Uparrow & \subseteq \ldots \subseteq & L_w^+\Uparrow
\end{array}$$

All these helpful facts we present in Figure 6.

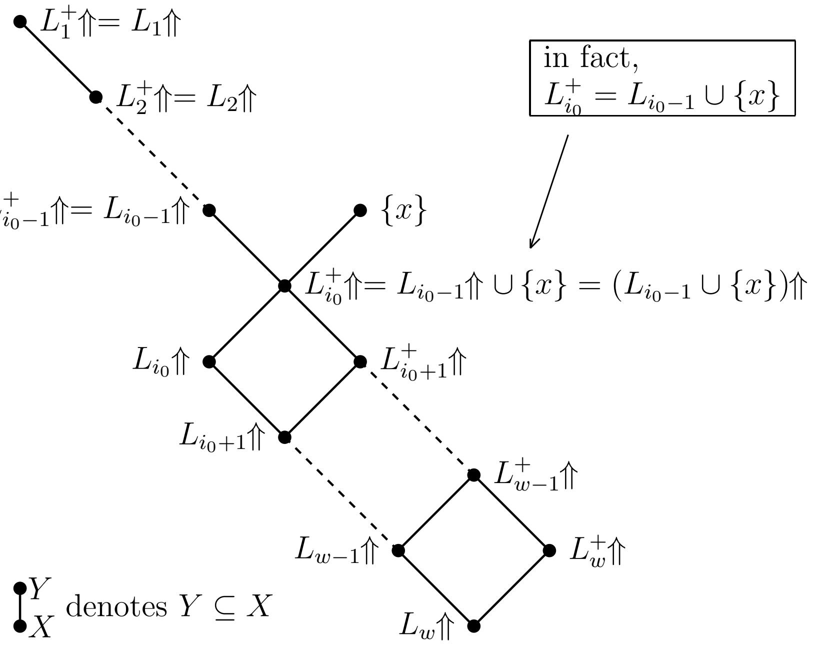

FIGURE 6

Proof. Now, we are going to prove all inclusions in the diagram above. From I1 for S it follows that $L_1\Uparrow \subseteq \ldots \subseteq L_w\Uparrow$. Moreover, we have already proved that $L_1^+\Uparrow \subseteq \ldots \subseteq L_w^+\Uparrow$. By (A16) we also know that $x \in L_{i_0}\uparrow$ and $x \notin L_{i_0-1}\uparrow$.

Directly from the definition of L_i^+(see (A19)) we obtain that $L_i^+\Uparrow \subseteq L_i\Uparrow$ for $i = 1, \ldots, w$. The fact that $L_i^+\Uparrow \supseteq L_i\Uparrow$ for $i \leqslant i_0 - 1$ does not influence our considerations but it is really helpful to understand the dynamic construction of levels of posets P and P^+. We leave it without proof. To get all other relations in the diagram all we need to prove is:

$$L_{i_0}^+ = L_{i_0-1} \cup \{x\}\,. \tag{4}$$

Since $L_{i_0}^+ = HMA(L_{i_0}\Uparrow)$ it suffices to prove two things: $L_{i_0-1} \cup \{x\} \subseteq L_{i_0}\Uparrow$ and $L_{i_0-1} \cup \{x\}$ is high in P^+. The first relation we get from I1 for S and (A16). The second statement follows from the fact that L_{i_0-1} is high in P (see I1 for S) and that x is maximal in P^+. This proves all claims contained in Figure 6. □

To prove I3 for S^+ we will need that

$$L_i\Downarrow \subseteq L_i^+\Downarrow\,. \tag{5}$$

This easily follows from the following Remark applied to maximum (by (1)) antichains in $L_i\Uparrow$ satisfying $L_i^+\Uparrow\subseteq L_i\Uparrow$.

Remark. *Let A and B be maximum antichains in a subposet Q of $\mathcal{R}$. Then $A \subseteq B\Uparrow_{\mathcal{R}}$ iff $B \subseteq A\Downarrow_{\mathcal{R}}$.*

Proof. Since A and B are maximum antichains in Q we know that

$$Q \subseteq A\uparrow_{\mathcal{R}} \cup A\Downarrow_{\mathcal{R}} \qquad \text{and} \qquad Q \subseteq B\uparrow_{\mathcal{R}} \cup B\Downarrow_{\mathcal{R}} . \tag{6}$$

From $A \subseteq B\Uparrow_{\mathcal{R}}$ we get $A\uparrow_{\mathcal{R}}\subseteq B\uparrow_{\mathcal{R}}$. This together with (6) yields

$$B \subseteq B\Downarrow_{\mathcal{R}} \cap Q = Q - B\uparrow_{\mathcal{R}}\subseteq Q - A\uparrow_{\mathcal{R}}\subseteq A\Downarrow_{\mathcal{R}} .$$

The converse follows by symmetry. □

Claim. $\alpha_i^+ \subseteq L_i^+\Downarrow - L_{i-1}^+\Uparrow$, $\beta_i^+ \subseteq L_i^+\downarrow$.

Proof. The key observations used during this proof are (5) and claims included in Figure 6. First we prove I3 for $i \neq i_0$. Thus, consider $\alpha_i^+ = \alpha_i$ and $\beta_i^+ = \beta_i$ (see (A20),(A21)). By I3 for S we have

$$\alpha_i^+ = \alpha_i \subseteq (L_i\Downarrow - L_{i-1}\Uparrow) \cap P \subseteq L_i\Downarrow - L_{i-1}\Uparrow, \quad \text{for } i \neq i_0,$$

and

$$\beta_i^+ = \beta_i \subseteq L_i\downarrow \cap P = L_i\downarrow, \quad \text{for } i \neq i_0.$$

Now, applying $L_i^+\Uparrow\subseteq L_i\Uparrow$ (see Figure 6) and $L_i\Downarrow\subseteq L_i^+\Downarrow$ (see (5)) we obtain

$$\alpha_i^+ \subseteq L_i\Downarrow - L_{i-1}\Uparrow\subseteq L_i^+\Downarrow - L_{i-1}^+\Uparrow$$

and

$$\beta_i^+ \subseteq L_i\downarrow\subseteq L_i^+\downarrow,$$

which finishes the proof for $i \neq i_0$.

Now we consider the chain $\beta_{i_0}^+ = \alpha_{i_0}$ (see (A21)). By I3 for S we have $\alpha_{i_0} \subseteq L_{i_0}\Downarrow - L_{i_0-1}\Uparrow$ and therefore

$$\beta_{i_0}^+ = \alpha_{i_0} \subseteq L_{i_0}\Downarrow\subseteq L_{i_0}^+\Downarrow . \tag{7}$$

Again, using I3 for S we get that $\beta_{i_0}^+$ is disjoint with $L_{i_0-1}\Uparrow$. Of course, $x \notin \alpha_{i_0}$. Thus $\beta_{i_0}^+$ is disjoint with $L_{i_0}^+ = L_{i_0-1} \cup \{x\}$. Getting this together with (7) we obtain I3 for $\beta_{i_0}^+$.

It remains to prove I3 for $\alpha_{i_0}^+ = \beta_{i_0} \cup \{x\}$. For the old part β_{i_0} of $\alpha_{i_0}^+$ we have

$$\begin{aligned} \beta_{i_0} &\subseteq L_{i_0}\downarrow \cap P && \text{by I3 for } S \\ &= L_{i_0}\Downarrow - L_{i_0}\Uparrow && \text{by } x \in L_{i_0}\uparrow \text{ (see (A16))} \\ &\subseteq L_{i_0}\Downarrow - L_{i_0-1}\Uparrow . && \text{by I1 for } S \text{ or Figure 6} \end{aligned}$$

This together with $L_{i_0}^+\Uparrow\subseteq L_{i_0}\Uparrow$ and $L_{i_0}\Downarrow\subseteq L_{i_0}^+\Downarrow$ gives

$$\beta_{i_0} \subseteq L_{i_0}\Downarrow - L_{i_0-1}\Uparrow\subseteq L_{i_0}^+\Downarrow - L_{i_0-1}^+\Uparrow .$$

It remains to show that also $x \in L_{i_0}^+\Downarrow - L_{i_0-1}^+\Uparrow$. But this follows from (4), the minimality condition (A16) and the fact that $L_{i_0-1}^+\Uparrow\subseteq L_{i_0-1}\Uparrow$. □

We have just shown that both in Case A and B the conditions I1-I3 are kept invariant by the algorithm when expanding S to S^+. In particular I2 tells us that, in every moment the poset is covered by at most $2w$ chains, namely $\alpha_1, \ldots, \alpha_w, \beta_1, \ldots, \beta_w$.

A careful inspection of the algorithm allows us to eliminate β_1. Indeed, everytime algorithm uses β_1 (Case B: $i_0 = 1$) to cover a new point x it could use chain α_1 as well. It immediately follows from the fact that in this setting $x \in L_1\Uparrow$ and $\alpha_1 \subseteq L_1\Downarrow$. □

Acknowledgment. We want to thank Paweł Idziak for many hours spent with us and Stefan Felsner for his much appreciated help.

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