TOWARDS ON-LINE OHBA'S CONJECTURE

JAKUB KOZIK, PIOTR MICEK, AND XUDING ZHU

ABSTRACT. Ohba conjectured that every graph G with $|V(G)| \leqslant 2\chi(G)+1$ has its choice number equal its chromatic number. The on-line choice number of a graph is a variation of the choice number defined through a two person game, and is always at least as large as its choice number. Based on the result that for $k \geqslant 3$, the complete multipartite graph $K_{2\star(k-1),3}$ is not on-line k-choosable, the on-line version of Ohba's conjecture is modified in [P. Huang, T. Wong and X. Zhu, Application of polynomial method to on-line colouring of graphs, European J. Combin., 2011] as follows: Every graph G with $|V(G)| \leqslant 2\chi(G)$ has its on-line choice number equal its chromatic number. In this paper, we prove that for any graph G, there is an integer n such that the join $G + K_n$ of G and K_n has its on-line choice number equal chromatic number. Then we show that the on-line version of Ohba conjecture is true if G has independence number at most 3. We also present an alternative proof of the result that Ohba's conjecture is true for graphs of independence number at most 3 and an alternative proof of the following result of Kierstead: For any positive integer k, the complete multipartite graph $K_{3\star k}$ has choice number $\lceil (4k-1)/3 \rceil$. Finally, we prove that the on-line choice number of $K_{3\star k}$ is at most $\frac{3}{2}k$. The exact value of the on-line choice number of $K_{3\star k}$ remains unknown.

1. INTRODUCTION

A *list assignment* of a graph G is a mapping L which assigns to each vertex v a set $L(v)$ of permissible colours. An *L-colouring* of G is a proper vertex colouring of G which colours each vertex with one of its permissible colours. We say that G is *L-colourable* if there exists an L-colouring of G. A graph G is called k-choosable if for any list assignment L with $|L(v)| = k$, for all $v \in V(G)$, G is L-colourable. More generally, for a function $f : V(G) \to \mathbb{N}$, we say G is *f-choosable* if for every list assignment L with $|L(v)| = f(v)$, G is L-colourable. The *choice number* $\mathrm{ch}(G)$ of G is the minimum k for which G is k-choosable. List colouring of graphs has been studied extensively in the literature [21, 3, 20].

A list assignment of a graph G can be given alternatively as follows: Without loss of generality, we may assume that $\cup_{v\in V(G)} L(v) = \{1, 2, \ldots, q\}$ for some integer q. For $i = 1, 2, \ldots, q$, let $V_i = \{v : i \in L(v)\}$. The sequence $(V_1, V_2, \ldots, V_q)$ is another way of specifying the list assignment. An L-colouring of G is equivalent to a sequence $(X_1, X_2, \ldots, X_q)$ of independent sets that form a partition of $V(G)$ and such that $X_i \subseteq V_i$ for $i = 1, 2, \ldots, q$. This point of view of list colouring motivates the definition of the following list colouring game on a graph G, which was introduced in [18, 17].

Definition. *Given a finite graph G and a mapping $f : V(G) \to \mathbb{N}$, two players play the following game. In the i-th step, Player A chooses a non-empty subset V_i of $V(G)$, and Player B chooses an independent set X_i contained in V_i. A vertex v is*

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coloured *before the ith step if* $v \in X_j$ *for some* $j < i$*, and is* finished *before the ith step if* v *is contained in* $f(v)$ *of the* V_j*'s with* $j < i$*. When Player A chooses the set* V_i*, it is required* V_i *contains only uncoloured non-finished vertices. If for some integer* m*, before the* m*-th step, there is a finished vertex* v *that is uncoloured, then Player A wins the game. Otherwise, at some step, all vertices are coloured. In this case, Player B wins the game.*

We call such a game the *on-line* (G, f)*-list colouring game*. We say G is *on-line* f*-choosable* if Player B has a winning strategy in the on-line (G, f)-list colouring game, and we say G is *on-line* k*-choosable* if G is on-line f-choosable for the constant function $f \equiv k$. The *on-line choice number* of G, denoted by $\mathrm{ch}^{\mathrm{OL}}(G)$, is the minimum k for which G is on-line k-choosable.

It follows from the definition that for any graph G, $\mathrm{ch}^{\mathrm{OL}}(G) \geqslant \mathrm{ch}(G)$. There are graphs G with $\mathrm{ch}^{\mathrm{OL}}(G) > \mathrm{ch}(G)$ (see [22]). It remains a challenging open problem whether the difference $\mathrm{ch}^{\mathrm{OL}}(G) - \mathrm{ch}(G)$ can be arbitrarily large. Alon [1] proved that if $\mathrm{ch}(G) \leqslant k$ then its colouring number $\mathrm{col}(G)$ is at most $f(k) = 4\binom{k^4}{s}\log_2(2\binom{k^4}{k})$. This gives us an exponential bound for the on-line choice number of G in terms of its choice number

$$f(\mathrm{ch}(G)) \geqslant \mathrm{col}(G) \geqslant \mathrm{ch}^{\mathrm{OL}}(G).$$

Many currently known upper bounds for the choice number of a graph remain upper bounds for its on-line choice number. For example, the on-line choice number of planar graphs is at most 5 [17], the on-line choice number of planar graphs of girth at least 5 is at most 3 [17, 2], the on-line choice number of the line graph $L(G)$ of a bipartite graph G is $\Delta(G)$ [17], and if G has an orientation in which the number of even eulerian subgraphs differs from the number of odd eulerian subgraphs and $f(x) = d^+(x) + 1$, then G is on-line f-choosable [18].

A graph G is called *chromatic-choosable* (respectively, *on-line chromatic-choosable*) if $\chi(G) = \mathrm{ch}(G)$ (respectively, $\chi(G) = \mathrm{ch}^{\mathrm{OL}}(G)$). The problem which graphs are chromatic-choosable has been extensively studied. A few well-known classes of graphs are conjectured to be chromatic-choosable. These include line graphs (conjectured independently by Vizing, by Gupta, by Albertson and Collins, and by Bollobás and Harris, see [6] and [9]), claw-free graphs [5], and square of graphs [13], etc. It is proved by Galvin [4] that the line graph of a bipartite graph is always chromatic-choosable. As observed by Schauz [17], the same proof works for on-line list colouring as well. So the line graph of a bipartite graph is on-line chromatic-choosable. In this paper, we are interested in Ohba's conjecture [14], which also concerns chromatic-choosable graphs.

Conjecture 1 (Ohba 2002)**.** *If* $|V(G)| \leqslant 2\chi(G) + 1$*, then* $\chi(G) = \mathrm{ch}(G)$*.*

Some special cases of Ohba's conjecture are already verified. Reed and Sudakov [16, 15] proved that it holds for all graphs G with $|V(G)| \leqslant \frac{5}{3}\chi(G) - \frac{4}{3}$ and soon afterwards they gave an asymptotic-type result that for any $\varepsilon > 0$ there is an integer n_0 such that all graphs with $n_0 \leqslant |V(G)| \leqslant (2 - \varepsilon)\chi(G)$ are chromatic-choosable. Recently, Kostochka et al. (see [12]) proved that Conjecture 1 holds for all graphs with independence number at most 5 which improves the results of [7, 19].

Note that it suffices to consider the conjecture only for complete multipartite graphs. Suppose $k = k_1 + k_2 + \ldots + k_s$, and $n_1, n_2, \ldots, n_s$ are positive integers. We denote by $K_{n_1\star k_1, n_2\star k_2, \ldots, n_s\star k_s}$ the complete k-partite graph in which k_i parts are of cardinality n_i for $i = 1, 2, \ldots, s$. If $k_i = 1$, then $n_i \star 1$ in the subscript will be shortened as n_i (for example $K_{3,2\star 3} = K_{3\star 1, 2\star 3}$).

It is proved in [11] that for $k \geqslant 2$, the graph $K_{3,2\star k}$ is not on-line $(k+1)$-choosable. However, experiments and preliminary results show that a slightly modified version of Ohba's conjecture might be true in the on-line case. The following conjecture is proposed in [8].

Conjecture 2. *If* $|V(G)| \leqslant 2\chi(G)$*, then* $\chi(G) = \mathrm{ch}^{\mathrm{OL}}(G)$*.*

The on-line version of Ohba's conjecture seems to be more difficult to handle. Some of the key technique used in the study of Ohba's conjecture do not apply to the on-line version. For example, it is easy to prove that $K_{2\star k}$ is k-choosable. However, all the previously known proofs of this result use Hall Theorem, and this cannot be directly applied to the on-line version. In [8], the method of Combinatorial Nullstellensatz is used to prove that $K_{2\star k}$ is k-choosable. By a result of Schauz mentioned above, this implies that $K_{2\star k}$ is on-line k-choosable. Recently, a simple strategy was given in [11] for Player B to win this on-line (G, f)-colouring game.

By using Combinatorial Nullstellensatz, $K_{\ell+1,1\star(\ell-1),2\star(k-\ell)}$, $K_{s,t,1\star(k-2)}$ (where $(s-1)(t-1) \leqslant 2k-3$), $K_{3\star2,1\star2,2\star(k-4)}$ and $K_{4,3,1\star3,2\star(k-5)}$ are shown in [8] to be on-line k-choosable. Still, we know much less about Conjecture 2 than about Conjecture 1.

The main focus of this paper is the on-line version of Ohba's conjecture. We prove that for any graph G, by adding enough universal vertices, the resulting graph is on-line chromatic-choosable. I.e., for a sufficiently large integer n, the join $G + K_n$ of G and K_n is on-line chromatic-choosable. In fact the argument gives that $\chi(G) = \mathrm{ch}^{\mathrm{OL}}(G)$ for all graphs G with $|V(G)| \leqslant \chi(G) + \sqrt{\chi(G)}$. Then we prove that Conjecture 2 is true for graphs with independence number at most 3, and also give an alternate proof of the result that Conjecture 1 is true for graphs with independence number at most 3.

We finish with the discussion on the choice number and on-line choice number of $K_{3\star k}$. These graphs are natural candidates to prove a hypothetic separation (by more than a constant) of choice number and on-line choice number. With an ingenious argument, Kierstead proved in [10] that $\mathrm{ch}(K_{3\star k}) \leqslant \lceil (4k-1)/3 \rceil$. This result matches the lower bound given by Erdös, Rubin and Taylor [3]. We prove that $\mathrm{ch}^{\mathrm{OL}}(K_{3\star k}) \leqslant \frac{3}{2}k$, and present an alternative proof of Kierstead's result.

2. The join of G and K_n

We are going to prove here that for any graph G, by adding enough universal vertices, one can construct a graph that is on-line chromatic-choosable. For two graphs G and G', the *join* of G and G', denoted by $G + G'$ is the graph obtained from the disjoint union of G and G' by adding all the possible edges between $V(G)$ and $V(G')$.

Theorem 3. *For every graph* G *there exists a positive integer* n *such that* $\chi(G + K_n) = \mathrm{ch}^{\mathrm{OL}}(G + K_n)$*.*

Proof. Without loss of generality, we may assume that G is a complete $\chi(G)$-partite graph. Let us start with an easy observation (see [17]): Assume H is a graph and $f : V(H) \to \mathbb{N}$ is a function. If $f(v) \geqslant d_H(v) + 1$ for a vertex $v \in V(H)$, then

$$H \text{ is on-line } f\text{-choosable if and only if } H - v \text{ is on-line } f\text{-choosable.}$$

For a given graph G, we put $H_0 = G + K_n$ with $n = |V(G)|^2$ and $f(v) = \chi(H_0) = \chi(G) + n$ for all $v \in V(H_0)$. Let $V_1, \ldots, V_{\chi(H_0)}$ be a partition of $V(H_0)$ into independent sets. We are going to present a winning strategy for Player B in the on-line (H_0, f)-list colouring game.

We denote by H_i a subgraph of all uncoloured vertices of H_0 after i steps. Before playing the $(i+1)$-th step, we delete from H_i, one by one, all the vertices v with $f(v) \geqslant d_{H_i}(v) + 1$ (by using the observation above). The resulting graph is still denoted by H_i. Now, by a *part* of H_i we mean a non-empty set of the form $V_j \cap H_i$ for $1 \leqslant j \leqslant \chi(H_0)$. Assume at the $(i+1)$th step, Player A chooses a subset U_i. Player B finds an independent set I contained in U_i according to the following algorithm.

Algorithm 1: Strategy for Player B in the $(i+1)$-th step

1 **if** there is a part V of H_i with $|V| \geqslant 2$ and $V \subseteq U_i$ **then**
2   **pick** $I = V$
3 **else if** there is a part V of H_i with $|V| = 1$ and $V \subseteq U_i$ **then**
4   **pick** $I = V$
5 **else**
6   **pick** I to be any maximal independent set in U_i

For $v \in V(G \cap H_i)$, let $f_i(v)$ be the number of remaining colours for v just before the $(i+1)$th step, and define the *deficit* of v as $d_{H_i}(v) + 1 - f_i(v)$, which is the number of additional colours needed so that v can be removed from the graph (by the observation we started with). Since vertices v with $f_i(v) \geqslant d_{H_i}(v) + 1$ are removed, we know that the deficit of each vertex v is positive. The deficit of a part V of H_i is the sum of deficits of its vertices

$$\sum_{v \in V} (d_{H_i}(v) + 1 - f_i(v)).$$

We will show that after every step of the game the deficit of each part of size at least 2 decreases. Let V be a part of H_i and $|V| \geqslant 2$.

If line 2 is executed, then either part V is picked and it disappears in H_{i+1}, or $d_{H_{i+1}}(v) \leqslant d_{H_i}(v) - 2$ and $f_{i+1}(v) \geqslant f_i(v) - 1$ for all $v \in V$. Hence the deficit of each vertex of V decreases.

If line 4 is executed, then $d_{H_{i+1}}(v) = d_{H_i}(v) - 1$, $f_{i+1}(v) \geqslant f_i(v) - 1$ for all $v \in V$, and there exists $v \in V$ with $f_{i+1}(v) = f_i(v)$ as V is not contained in U_{i+1}. So the total deficit of V decreases.

Assume line 6 is executed. If $I = V \cap U_{i+1}$, then $d_{H_{i+1}}(v) = d_{H_i}(v)$, $f_{i+1}(v) = f_i(v)$ for all $v \in V - U_{i+1}$ so the sum decreases as the deficit of erased vertices is positive. Otherwise, $d_{H_{i+1}}(v) \leqslant d_{H_i}(v) - 1$ and $f_{i+1}(v) \geqslant f_i(v) - 1$ for all $v \in V$ and there exists $v \in V$ with $f_{i+1}(v) = f_i(v)$. So the deficit of V decreases.

As each $v \in V(H_0)$ has deficit bounded by $|V(G)|$, each part has initially deficit bounded by $n = |V(G)|^2$. Since after each step the deficit of each part of size at least 2 decreases and vertices with non-positive deficit are deleted, after n rounds the remaining graph, namely H_n, forms a clique.

The vertices in H_n may come from G or K_n and there are at most $\chi(G)$ vertices coming from G, at most one for each part of G. If $U_i \cap K_n \neq \emptyset$ then the number of parts in H_{i+1} decreases by 1 comparing to the number of parts in H_i (as line 2 or 4 is executed). Therefore

$$f_n(v) \geqslant \text{the number of parts in } H_n = d_{H_n}(v) + 1 \qquad \text{for all } v \in H_n \cap K_n$$

For vertices $v \in H_n \cap G$, as each step decreases the number of permissible colours by at most 1, we have $f_n(v) \geqslant f_0(v) - n = \chi(G)$. By applying the observation repeatedly, these inequalities certify that all vertices of H_q are removed and H_q is empty, which finishes the proof. □

The argument presented gives also an Ohba-like statement with much more restricted constraint on the size and the chromatic number of a graph.

Corollary 4. *If* $|V(G)| \leqslant \chi(G) + \sqrt{\chi(G)}$, *then* $\chi(G) = \mathrm{ch}^{\mathrm{OL}}(G)$.

3. A lemma

In the remainder of this paper, we consider complete multipartite graphs of independence number at most 3, i.e., graphs of the form $K_{3\star k_3,2\star k_2,1\star k_1}$ for some integers $k_1, k_2, k_3 \geqslant 0$. Lemma 5 below specifies a sufficient condition for such a graph G to be on-line f-choosable. In further sections we are going to derive from this a few quite independent results.

For a subset U of $V(G)$, let $\delta_U : V(G) \to \{0, 1\}$ be the characteristic function of U, i.e., $\delta(x) = 1$ if $x \in U$ and $\delta_U(x) = 0$ otherwise. The following observation follows directly from the definition of the on-line (G, f)-colouring game (see [17]).

Observation. *If G is an edgeless graph and $f(v) \geqslant 1$ for all $v \in V(G)$, then G is on-line f-choosable. If G has at least one edge, then G is on-line f-choosable if and only if for every $U \subseteq V(G)$, there is an independent set I of G such that $I \subseteq U$ and $G - I$ is on-line $(f - \delta_U)$-choosable.*

Lemma 5. *Let G be a complete multipartite graph G with each part of size at most 3. Let $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{S}$ be a partition of the set of parts of G into classes such that $\mathcal{A}$ contains only parts of size 1, $\mathcal{B}$ contains only parts of size 2, $\mathcal{C}$ contains only parts of size 3 and $\mathcal{S}$ contains parts of size 1 or 2. Let k_1, k_2, k_3, s denote the cardinalities of classes $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{S}$, respectively. Suppose that classes $\mathcal{A}$ and $\mathcal{S}$ are ordered i.e. $\mathcal{A} = (A_1, \ldots, A_{k_1})$ and $\mathcal{S} = (S_1, \ldots, S_s)$. For $1 \leqslant i \leqslant s$, let $v_S(i) = \sum_{1\leqslant j<i} |S_i| + 1$. Assume $f : V(G) \to \mathbb{N}$ is a function for which the following conditions hold*

$$f(v) \geqslant k_3 + k_2 + i, \qquad \text{for all } 1 \leqslant i \leqslant k_1 \text{ and } v \in A_i \tag{1}$$

$$f(v) \geqslant 2k_3 + k_2 + k_1 + v_S(i), \qquad \text{for all } 1 \leqslant i \leqslant s \text{ and } v \in S_i \tag{1'}$$

$$f(v) \geqslant k_3 + k_2, \qquad \text{for all } v \in B \in \mathcal{B} \tag{2.1}$$

$$\sum_{v\in B} f(v) \geqslant |V(G)|, \qquad \text{for all } B \in \mathcal{B} \tag{2.2}$$

$$f(v) \geqslant k_3 + k_2, \qquad \text{for all } v \in C \in \mathcal{C} \tag{3.1}$$

$$f(u) + f(v) \geqslant |V(G)| - 1, \qquad \text{for all } u, v \in C \in \mathcal{C},\ u \neq v \tag{3.2}$$

$$\sum_{v\in C} f(v) \geqslant |V(G)| - 1 + k_3 + k_2 + k_1, \qquad \text{for all } C \in \mathcal{C}. \tag{3.3}$$

Then G is on-line f-choosable.

Proof. The proof goes by induction on $|V(G)|$. If G is edgeless, i.e., $k_1+k_2+k_3+s = 1$, then G is on-line f-choosable as $f(v) \geqslant 1$ for all $v \in V(G)$. Assume now that G has at least two parts and that the statement is verified for all smaller graphs.

Given $U \subseteq V(G)$, we shall find an independent set I of G such that $I \subseteq U$ and $G - I$ is on-line $(f - \delta_U)$-choosable. Let $G' = G - I$ and $f' = f - \delta_U$. Note that $f'(v) \geqslant f(v) - 1$ for all $v \in V(G)$. Clearly, G' is also a complete multipartite graph with each part of size at most 3. We are going to show that G' with f', an appropriate partition $\mathcal{A}', \mathcal{B}', \mathcal{C}', \mathcal{S}'$ and orderings of $\mathcal{A}'$ and $\mathcal{S}'$ fulfill the conditions of Lemma 5. Hence, by induction hypothesis G' is on-line f'-choosable.

The strategy of choosing an independent set I is given by the case distinction. Note that we consider the setting of Case i only when the conditions for all $i - 1$ previous cases do not hold. When we verify the inequalities from the statement of Lemma 5 for G' and f' we usually compare the total decrease/increase of left and right hand sides with the analogous inequalities that hold for G and f. The notation for the parts of G' and its sizes is analogous as for G, e.g. A_i', S_i', k_1', s' and so on. Partition $\mathcal{A}', \mathcal{B}', \mathcal{C}', \mathcal{S}'$ and orders on the classes $\mathcal{A}'$ and $\mathcal{S}'$ are usually inherited. In the case distinction below we comment the partitions only if the order or partition changes in the considered step.

Case 1. $C \subseteq U$ for some $C \in \mathcal{C}$.

Put $I = C$. Then $k_3' = k_3 - 1$ and all other parameters remain the same. Note that $|V(G')| = |V(G)| - 3$. Now, it is immediate that G' with inherited partition and f' satisfy the conditions of Lemma 5.

Case 2. $B \subseteq U$ for some $B \in \mathcal{B}$.

Put $I = B$. Then $k_2' = k_2 - 1$ and all other parameters remain the same. Note that $|V(G')| = |V(G)| - 2$. Again, it is immediate that G' with inherited partition and f' satisfy the conditions of Lemma 5. Note that because Case 1 does not apply, for inequality (3.3), the left-hand side decreases by at most 2.

In all remaining cases, as conditions for cases 1 and 2 do not hold, we have

(i) U covers at most one vertex in each $B \in \mathcal{B}$ (we are not in Case 2). This implies that inequalities (2.2) for any G' will trivially hold provided $|V(G')| \leqslant |V(G)| - 1$.
(ii) U covers at most two vertices in each $C \in \mathcal{C}$ (we are not in Case 1).

Case 3. There is $C \in \mathcal{C}$ with $U \cap C = \{u, v\}$ and (3.1) is saturated for v or (3.2) is saturated for u and v.

Let $C = \{u, v, w\}$. Put $I = \{u, v\}$. Then $k_3' = k_3 - 1$, $k_1' = k_1 + 1$ and all other parameters remain unchanged. Indeed, we colour two vertices of C and the remaining vertex forms $A'_{k_1'} = \{w\}$, a new part of size 1, which is appended to the ordering of $\mathcal{A}'$. Note that $|V(G')| = |V(G)| - 2$.

Now, we need to check that all the inequalities of Lemma 5 hold for G' and f'. Inequality (1) holds for A_i' with $1 \leqslant i < k_1'$ as the right hand side decreases by 1 and the left hand side decreases at most by 1. Inequality (1) holds for $A'_{k_1'} = \{w\}$ either because (3.2) is saturated for u, v in G and hence

$$f'(w) = f(w) \geqslant |V(G)| - 1 + k_3 + k_2 + k_1 - (|V(G)| - 1) = k_3' + k_2' + k_1',$$

or because (3.1) is saturated for v in G and hence

$$f'(w) = f(w) \geqslant |V(G)| - 1 - k_3 - k_2 \geqslant 2k_3 + k_2 + k_1 - 1 = 2(k_3' + 1) + k_2' + (k_1' - 1) - 1.$$

The inequality (3.3) for $C \in \mathcal{C}$ holds as the right hand side decreased by 2 and the left hand side decreased by at most 2 (see (ii)). The other inequalities hold trivially.

Note that in all remaining cases

(i) For each $C \in \mathcal{C}$ either $|U \cap C| \leqslant 1$, or $|U \cap C| = 2$ and (3.2) is not saturated for $U \cap C$ in G (we are not in Case 3). This implies that inequalities (3.2) will hold for any G' provided $|V(G')| \leqslant |V(G)| - 1$.

Case 4. There is $B \in \mathcal{B}$ with $U \cap B = \{v\}$ and (2.1) is saturated for v.

Let $B = \{u, v\}$. Put $I = \{v\}$. Then $k_2' = k_2 - 1$ and $s' = s + 1$ and all other parameters remain unchanged. The part $\{u\}$ form a new part of size 1 and is appended at the end of the order to the class $\mathcal{S}$ as $S'_{s'}$. Note that $|V(G')| = |V(G)| - 1$.

We are going to check the inequalities for G' and f'. Inequalities (1) for A_j' with $1 \leqslant j \leqslant k_1'$ and (1') for S_j' with $1 \leqslant j \leqslant s' - 1$ hold as the right hand side decreases by 1 while the left hand side decreases at most by 1. Inequality (1') for $S'_{s'} = \{u\}$ holds by (2.2) for u, v in G and the saturation of (2.1) for v in G

$$f'(u) = f(u) \geqslant |V(G)| - k_3 - k_2 = 2k_3' + (k_2' + 1) + k_1 + (v_{S'}(s') - 1).$$

The inequalities (2.1), (3.1) and (3.3) for G' with f' hold trivially.

Note that in all remaining cases

(i) For all $v \in U \cap \bigcup_{B \in \mathcal{B}} B$ the inequality (2.1) is not saturated for v in G. This means that (2.1) will hold in any G'.

Case 5. There is $C \in \mathcal{C}$ with $U \cap C = \{v\}$ and (3.1) is saturated for v.

Let $C = \{u, v, w\}$ and put $I = \{v\}$. The remaining part $\{u, w\}$ is appended at the end of the sequence $\mathcal{S}$. Note that $|V(G')| = |V(G)| - 1$.

The inequalities (1) for A'_j with $1 \leqslant j \leqslant k'_1$ and (1') for S'_j with $1 \leqslant j \leqslant s-1$ hold as the right hand side decreases by 1 while the left hand side decreases at most by 1. Inequalities (1') for the vertices of the new part $S'_{s'} = \{u, w\}$ hold because (3.1) is saturated for v in G and hence for $x \in \{u, w\}$,

$$f'(x) = f(x) \geqslant (|V(G)| - 1) - k_3 - k_2 = 2(k'_3 + 1) + k'_2 + k'_1 + (v_{S'}(s') - 1) - 1,$$

The inequalities (2.1), (3.1) for G' are trivial. The inequalities (3.3) for G' hold as the right hand side decreases by 2 and the left hand side at most by 2 (see (ii)).

Note that in all remaining cases

(i) For all $v \in U \cap \bigcup_{C \in \mathcal{C}} C$ the inequality (3.1) is not saturated for v in G. This means that (3.1) will hold in any G'.

Case 6. There is $1 \leqslant i \leqslant k_1$ with $A_i \subseteq U$.

Let i be the least index with $A_i = \{v\} \subseteq U$. Put $I = \{v\}$. Then $k'_1 = k_1 - 1$ and all other parameters remain unchanged. We also renumber the parts of size 1, namely $A'_j = A_{j+1}$ for $i \leqslant j \leqslant k'_1$. Note that $|V(G')| = |V(G)| - 1$.

The inequality (1) for A'_j with $1 \leqslant j < i$ holds as both sides are the same in G' as in G. The inequality (1) for $A'_j = A_{j+1} = \{u\}$ with $i \leqslant j \leqslant k'_1$ holds as

$$f'(u) \geqslant f(u) - 1 \geqslant k_3 + k_2 + (j + 1) - 1.$$

The inequalities (3.3) hold in G' as the right hand side decreased by 2 and the left hand side decreased by at most 2 (see (ii)).

Case 7. There is $C \in \mathcal{C}$ with $U \cap C = \{u, v\}$.

Let $C = \{u, v, w\}$. Put $I = \{u, v\}$. Then $k'_3 = k_3 - 1$ and $k'_1 = k_1 + 1$ and all other parameters remain unchanged. There is one new part of size 1, namely $A'_1 = \{w\}$, and all the others are renumbered $A'_j = A_{j-1}$ for $2 \leqslant j \leqslant k'_1$. Note that $|V(G')| = |V(G)| - 1$.

The inequality (1) for $A'_1 = \{w\}$ holds by (3.1) for w in G

$$f'(w) = f(w) \geqslant k_3 + k_2 = (k'_3 + 1) + k'_2.$$

The inequality (1) for $A'_j = A_{j-1} = \{x\}$ with $2 \leqslant j \leqslant k'_1$ holds as $x \notin U$ (Case 6 does not apply)

$$f'(x) = f(x) \geqslant k_3 + k_2 + (j - 1) = k'_3 + k'_2 + j.$$

The inequalities (3.3) hold in G' as the right hand side decreased by 2 and the left hand side decreased by at most 2.

Note that in all remaining cases

(i) $|U \cap C| \leqslant 1$, for $C \in \mathcal{C}$. As we always have $|V(G')| \leqslant |V(G)| - 1$ and $k'_3 + k'_2 + k'_1 \leqslant k_3 + k_2 + k_1$ the inequalities (3.3) will hold for any G'.

Case 8. There is $1 \leqslant i \leqslant s$ with $S_i \cap U \neq \emptyset$.

Let i be the least i with $S_i \cap U \neq \emptyset$. Put $I = S_i \cap U$. Then $s' = s - 1$ and all other parameters remain unchanged. If $|S_i \cap U| = S_i$, we update the order of the parts in the sequence $\mathcal{S}$, in the following way, for $i \leqslant j \leqslant s'$ we put $S'_j = S_{j+1}$. If $|S_i \cap U| \neq S_i$ the order remains the same. Note that $|V(G')| \leqslant |V(G)| - 1$.

The inequalities (1) for A'_j with $1 \leqslant j \leqslant k'_1$ and (1') for S'_j with $1 \leqslant j < i$ hold as both sides does not change. For every vertex from parts $S_{j+1}, \ldots S_s$ the right hand side of the inequality (1') decreases by at least one, therefore inequalities hold. For the vertices from $S_i \setminus U$ (this set may be empty) both sides of inequality does not change, therefore inequality holds as before.

Note that in all remaining cases

(i) Inequalities (1) and (1') will hold in any G', provided that the right hand side does not increase.

Case 9. There is $C \in \mathcal{C}$ with $C \cap U \neq \emptyset$.

As Case 7 does not apply, $|C \cap U| = 1$. We put $I = C \cap U$. Say that $C \setminus U = \{u, v\}$ then we put $\{u, v\}$ into class $\mathcal{B}'$. It is straightforward that vertices from $\{v, u\}$ satisfy (2.1). They also satisfy (2.2) as

$$f'(u) + f'(v) = f(u) + f(v) \geqslant |V(G)| - 1 = |V(G')|.$$

Case 10. There is $B \in \mathcal{B}$ with $B \cap U \neq \emptyset$.

We put $I = B \cap U$. Say that $B \setminus U = \{u\}$. We put $\{u\}$ to the very beginning of the class $\mathcal{A}'$. By the observations above, all the inequalities hold, and hence G' is on-line f'-choosable (note that for Inequalities (1) and (1'), the right hand side does not increase, as k_2' decreases by 1 and the index increases by 1).

It is easy to see that one of the 10 cases above occurs and hence G is on-line f-choosable. □

4. Graphs with independence number 3

Theorem 6. *If G is a graph with independence number at most 3 and $|V(G)| \leqslant 2\chi(G)$, then $\chi(G) = \mathrm{ch}^{\mathrm{OL}}(G)$.*

Proof of Theorem 6. Without loss of generality, we can assume that G is a complete multipartite graph with parts of size at most 3. We are going to verify that G satisfies Lemma 5 with $\mathcal{S} = \emptyset$, $f \equiv \chi(G)$ and arbitrary order on the class $\mathcal{A}$ (when $\mathcal{S} = \emptyset$ the remaining classes of the partition are determined). Let k_1, k_2, k_3 denote the sizes of parts of sizes 1,2,3, respectively.

Inequalities for the single vertices (1), (2.1), (3.1) hold as $f(v) = \chi(G) = k_1 + k_2 + k_3$. Condition on pairs of vertices (2.2), (3.2) hold since $f(u) + f(v) = 2\chi(G) \geqslant |V(G)|$ (by the assumption on G). Moreover adding $\chi(G) = k_3 + k_2 + k_1$ on both sides of the inequality (3.2) gives (3.3).

Now, by Lemma 5 G is on-line chromatic-choosable. □

It was shown in [19] that Conjecture 1 is true for graphs with independence number at most 3. The proof is a little complicated. Next we give an alternative proof of this result. We shall need the following lemma proved in [10] and [16].

Lemma 7. *A graph G is k-choosable if it is L-colourable for every k-list assignment L such that $|\bigcup_{v \in V} L(v)| < |V|$.*

Theorem 8. *If G is a graph with independence number at most 3 and $|V(G)| \leqslant 2\chi(G) + 1$, then $\chi(G) = \mathrm{ch}(G)$.*

Proof. For a contradiction let G be a counterexample with minimum number of vertices. Let L be a $\chi(G)$-list assignment such that G is not L-colourable. By Theorem 6, we may assume that $|V(G)| = 2\chi(G) + 1$ and by Lemma 7 we assume that the number of colours occurring on all the list is at most $2\chi(G)$.

We can also assume that for every part $\{u, v\}$ of size 2 the lists $L(u)$ and $L(v)$ are disjoint. If not, then we pick a colour $c \in L(u) \cap L(v)$ and use it to colour both vertices. The remaining graph $G' = G - \{u, v\}$ still satisfies $|V(G')| \leqslant 2\chi(G') + 1$. Now, if G' is chromatic-choosable then G' is colourable from $L - \{c\}$. But this would imply that G is colourable from L. Thus G' is not chromatic-choosable which is a contradiction with the minimality of G. For the very same reason there is no colour that belongs to all three lists of vertices of any part of size 3 in G.

As $|V(G)| = 2\chi(G)+1$ there exists at least one part of size 3 in G, say $\{u, v, w\}$. Each vertex has a list of size $\chi(G)$ and the total number of colours is at most $2\chi(G)$, therefore there exists a colour c which belongs to lists of two vertices from this part, say $c \in L(u) \cap L(w)$.

We are going to construct an L-colouring of G in two steps. First, we use c to colour u and w, remove them from G and remove colour c from all lists. Than we prove that the remaining graph $G' = G - \{u, w\}$ is on-line f'-choosable, where

$$f'(v) = \begin{cases} \chi(G) & \text{if } c \notin L(v), \\ \chi(G) - 1 & \text{if } c \in L(v). \end{cases}$$

In particular, G' can be coloured from $L - \{c\}$, which finishes the colouring of G and gives the final contradiction.

The only thing we need to verify is that G' and f' satisfy the assumptions of Lemma 5 with $\mathcal{S} = \emptyset$ and parts from $\mathcal{A}$ ordered in such a way that the part $\{v\}$ has the greatest index. Let $k_1, k_2, k_3, k_1', k_2', k_3'$ denote the numbers of parts of size 1, 2 and 3 in G and G', respectively. We have

$$k_1' = k_1 + 1, \qquad k_2' = k_2, \qquad k_3' = k_3 - 1.$$

Inequalities (2.1), (3.1) hold as for any x in part of size 2 or 3 in G'

$$f'(x) \geqslant \chi(G) - 1 \geqslant k_3 + k_2 - 1 = k_3' + k_2'$$

The part of size 1, say $\{x\}$, with index less than k_1' satisfies (1) as

$$f'(x) \geqslant \chi(G) - 1 = k_3' + k_2' + k_1' - 1.$$

The remaining part of size 1, namely $\{v\}$, satisfies (1) as $f(v) = \chi(G) = \chi(G')$ (as $c \notin L(v)$). Inequalities (2.2) hold since colour c belongs to the list of at most one vertex in every part of size 2 in G'. Therefore, for any $\{x, y\}$ part of size 2 in G' we have

$$f'(x) + f'(y) \geqslant 2\chi(G) - 1 = |V(G')| - 1.$$

It remains to verify inequalities (3.2) and (3.3). Let x, y, z be any three vertices forming a part of size 3 in G'. Then

$$f'(x) + f'(y) \geqslant 2\chi(G) - 2 = |V(G')| - 1,$$
$$f'(x) + f'(y) + f'(z) \geqslant 3\chi(G) - 2 = |V(G')| - 1 + k_3' + k_2' + k_1'.$$

The latter inequality follows from the fact c is not in all three $L(x)$, $L(y)$, $L(z)$. □

5. The complete multipartite graph $K_{3\star k}$

There are not many graphs for which the exact value of their choice numbers are known. The graphs $K_{3\star k}$ are among those few graphs G for which $\mathrm{ch}(G)$ are determined. In [10], Kierstead proved that $\mathrm{ch}(K_{3\star k}) = \lceil (4k-1)/3 \rceil$. In this section, we present an alternative proof of this result.

Theorem 9 (Kierstead 2000)**.** *For any positive integer k, $\mathrm{ch}(K_{3\star k}) = \lceil \frac{4k-1}{3} \rceil$.*

The lower bound $\mathrm{ch}(K_{3\star k}) \geqslant \lceil \frac{4k-1}{3} \rceil$ was given by Erdös, Rubin and Taylor [3]. As the proof is very short, we include it here for the convenience of the reader. Let $q = \lceil \frac{4k-1}{3} \rceil - 1$. Let A, B, C be disjoint colour sets with $|A| = \lfloor q/2 \rfloor$ and $|B| = |C| = \lceil q/2 \rceil$. Assume the parts of $K_{3\star k}$ are $V_i = \{x_i, y_i, z_i\}$ for $i = 1, 2, \ldots, k$. Let $L(x_i) = A \cup B, L(y_i) = B \cup C$ and $L(z_i) = A \cup C$. Then $|L(v)| \geqslant q$ for each vertex v, and if f is an L-colouring of $K_{3\star k}$, then f uses at least 2 colours on V_i, and hence the total number of used colours is at least $2k$. However, straightforward calculation shows that $|A \cup B \cup C| \leqslant 2k - 1$. Therefore $K_{3\star k}$ is not L-colourable and hence $\mathrm{ch}(K_{3\star k}) \geqslant q + 1 = \lceil \frac{4k-1}{3} \rceil$.

The inequality $\mathrm{ch}(K_{3\star k}) \leqslant \lceil \frac{4k-1}{3} \rceil$ is a straightforward consequence of the following lemma.

Lemma 10. *Let G be a complete multipartite graph with parts of size 1 and 3. Let $\mathcal{A}$, $\mathcal{S}$, $\mathcal{C}$ be a partition of the set of parts of G into classes such that $\mathcal{A}$ and $\mathcal{S}$ contains only parts of size 1, while $\mathcal{C}$ contains all parts of size 3. Let k_1, s, k_3 denote the cardinalities of classes $\mathcal{A}$, $\mathcal{S}$, $\mathcal{C}$, respectively. Suppose that class $\mathcal{A}$ and $\mathcal{S}$ are ordered, i.e. $\mathcal{A} = (A_1, \ldots, A_{k_1})$ and $\mathcal{S} = (S_1, \ldots, S_s)$. If $f : V(G) \to \mathbb{N}$ is a function for which the following conditions hold*

$$f(v) \geqslant k_3 + i, \quad \text{for all } 1 \leqslant i \leqslant k_1 \text{ and } v \in A_i \tag{1}$$

$$f(v) \geqslant 2k_3 + k_1 + i, \quad \text{for all } 1 \leqslant i \leqslant s \text{ and } v \in S_i \tag{1'}$$

$$f(v) \geqslant k_3, \quad \text{for all } v \in C \in \mathcal{C} \tag{3.1}$$

$$f(u) + f(v) \geqslant 2k_3 + k_1, \quad \text{for all } u, v \in C \in \mathcal{C} \tag{3.2}$$

$$\sum_{v \in C} f(v) \geqslant 4k_3 + 2k_1 + s - 1, \quad \text{for all } C \in \mathcal{C}, \tag{3.3}$$

then G is f-choosable.

Proof. Assume the lemma is not true. Let G be a multipartite graph with parts divided into $\mathcal{A}$, $\mathcal{S}$, $\mathcal{C}$, and let f be a function fulfilling the inequalities (1)-(3.3) while G is not f-choosable. Moreover, suppose G is a counterexample with the minimum possible number of vertices. By Lemma 7 there exists a list assignment $\{L(v)\}_{v \in V(G)}$ with each $|L(v)| = f(v)$ and $|\bigcup_{v \in V(G)} L(v)| \leqslant |V(G)| - 1 = 3k_3 + k_1 + s - 1$ such that G is not L-colourable.

The claims below prove a series of properties of G and list assignment L. In the arguments we often make use the minimality of G and consider some smaller graphs with modified list assignment. The modified graph will be denoted by G' and, unless otherwise stated, the classes of its vertices $\mathcal{A}'$, $\mathcal{S}'$ and $\mathcal{C}'$, together with orders on $\mathcal{A}'$ and $\mathcal{S}'$, are inherited from G. The parameters k_1', s', k_3' correspond to the analogous parameters of G'. The modified list assignment is going to be denoted by $L'(v)$ and $f'(v) = |L'(v)|$ for all $v \in V(G')$.

Claim 0. For any $C \in \mathcal{C}$ we have $\bigcap_{v \in C} L(v) = \emptyset$.

Proof. Suppose there is $C \in \mathcal{C}$ with $c \in \bigcap_{v \in C} L(v)$. We colour all vertices of C with c and consider the smaller graph $G' = G - C$ with list assignment $L'(v) = L(v) - \{c\}$. It is easy to verify that G' (with $\mathcal{A}'$, $\mathcal{S}'$, $\mathcal{C}'$ inherited from G) and f' satisfies the assumptions of the lemma. By the minimality of G, G' is L'-colourable. This implies that G is L-colourable, in contrary to our assumption. □

Claim 1. For any $u, v \in C \in \mathcal{C}$ if $f(u) + f(v) = 2k_3 + k_1$, then $L(u) \cap L(v) = \emptyset$.

Proof. Suppose that for some part $C = \{u, v, w\}$ we have $f(u) + f(v) = 2k_3 + k_1$ and there exist $c \in L(u) \cap L(v)$. Then we colour u and v with c, and consider the smaller graph $G' = G - \{u, v\}$ with lists $L'(x) = L(v) - \{c\}$ for all $x \in V(G')$. The partition $\mathcal{A}'$, $\mathcal{C}'$ is inherited from G and $\mathcal{S}' = (\{w\}, S_1, \ldots, S_s)$ has one more part, namely $\{w\}$, while all other parts have shifted index, i.e., $S_{i+1}' = S_i$ for $1 \leqslant i \leqslant s$. In particular, $k_1' = k_1$, $s' = s + 1$, $k_3' = k_3 - 1$. Note that the inequality (1') holds for $S_1' = \{w\}$ as

$$f'(w) = f(w) \geqslant (4k_3 + 2k_1 + s - 1) - (2k_3 + k_1) = 2k_3 + k_1 + s - 1 = 2k_3' + k_1' + 1,$$

and (1') holds for $S_{i+1}' = S_i = \{x\}$ for $1 \leqslant i \leqslant s$ as

$$f'(x) \geqslant f(x) - 1 \geqslant (2k_3 + k_1 + i) - 1 = 2k_3' + k_1' + i + 1.$$

Again, it is easy to verify that G' with f' satisfies the assumptions of the lemma. Hence G' is L'-colourable, implying that G is L-colourable, a contradiction. □

Claim 2. For any $v \in C \in \mathcal{C}$ we have $f(v) > k_3$, i.e., the inequality (3.1) is not tight.

Proof. In order to get a contradiction suppose that $\{v, u, w\} = C \in \mathcal{C}$ and $f(v) = k_3$. We separate the argument into two cases:

* $L(v) \cap (L(u) \cup L(w)) \neq \emptyset$. Without loss of generality, assume that $L(v) \cap L(u) \neq \emptyset$. Let $c \in L(v) \cap L(u)$. We colour u and v with c, and consider the smaller graph $G' = G - \{u, v\}$ with lists $L'(x) = L(x) - \{c\}$ for all $x \in V(G')$. The partition $\mathcal{S}'$, $\mathcal{C}'$ is inherited from G and $\mathcal{A}' = (A_1, \ldots, A_{k_1}, \{w\})$ has one more part, namely $\{w\}$, appended to the inherited ordering. In particular, $k_1' = k_1 + 1$, $s' = s$, $k_3' = k_3 - 1$. Note that the inequality (1) holds for $A'_{k_1'} = \{w\}$ as

$$f'(w) = f(w) > (2k_3 + k_1) - k_3 = k_3' + k_1'.$$

Let $x, y \in C \in \mathcal{C}'$. Inequality (3.2) for x and y hold as either $f(x)+f(y) > 2k_3+k_1$ and therefore

$$f'(x) + f'(y) \geqslant f(x) + f(y) - 2 > 2k_3 + k_1 - 2 = 2k_3' + k_1' - 1,$$

or $f(x) + f(y) = 2k_3 + k_1$ and therefore by Claim 1 $L(x)$ and $L(y)$ are disjoint.

$$f'(x) + f'(y) \geqslant f(x) + f(y) - 1 = 2k_3 + k_1 - 1 = 2k_3' + k_1'.$$

With these observations, it is easy to verify that G' with f' satisfies the assumptions of the lemma. Hence G' is L'-colourable and therefore G would be L-colourable, a contradiction.

* $L(v) \cap (L(u) \cup L(w)) = \emptyset$. Then by (3.3) and our assumption $f(v) = k_3$ we get that

$$f(u) + f(w) \geqslant (4k_3 + 2k_1 + s - 1) - k_3 = 3k_3 + 2k_1 + s - 1.$$

On the other hand the total number of colours is at most $3k_3 + k_1 + s - 1$ and as $L(v)$ is disjoint with $L(u) \cup L(w)$ we get $|L(u) \cup L(w)| \leqslant 2k_3 + k_1 + s - 1$. Combining the two inequalities above we obtain

$$|L(u) \cap L(w)| \geqslant k_3 + k_1.$$

We colour vertex v by any colour $c \in L(v)$. Then we consider graph $G' = G - \{v, u, w\} + \{x\}$, where x is a brand new vertex which is convenient to be seen as a merger of u and w. Let $L'(y) = L(y) - \{c\}$ for all $y \in V(G') - \{x\}$ and $L'(x) = L(u) \cap L(w)$. The partition $\mathcal{S}'$, $\mathcal{C}'$ is inherited from G and $\mathcal{A}' = (A_1, \ldots, A_{k_1}, \{x\})$ has one more part, namely $\{x\}$, appended to the inherited ordering. In particular, $k_1' = k_1 + 1$, $s' = s$, $k_3' = k_3 - 1$. Note that the inequality (1) holds for $A'_{k_1'} = \{x\}$ as

$$f'(x) = |L(u) \cap L(w)| \geqslant k_3 + k_1 = k_3' + k_1'.$$

The other inequalities for G' and f' hold for the same reasons as before. So G' is L'-colourable. We obtain an L-colouing of G, by colouring the vertices u and w with the colour of x and colouring v with c, a contradiction.

□

Claim 3. $k_1 = 0$.

Proof. Suppose that $k_1 \neq 0$. Then let $A_1 = \{v\}$. We colour v with any colour $c \in L(v)$ and consider the smaller graph $G' = G - \{v\}$ with lists $L'(x) = L(x) - \{c\}$ for all $x \in V(G')$. The partition $\mathcal{A}' = (A_2, \ldots, A_{k_1})$, $\mathcal{S}'$, $\mathcal{C}'$ is inherited from G.

Note that $\mathcal{A}'$ has one part less and $k_1' = k_1 - 1$, $s' = s$, $k_3' = k_3$. Now, we verify the inequalities (1)-(3.3) for G' and f':

* (1) holds as the indices of parts are decreased, i.e. $A_i' = A_{i+1}$ for $1 \leqslant i < k_1'$;
* (1') holds as k_1 decreases,
* (3.1) holds as, by Claim 2, it is not tight in G,
* (3.2) holds for $x, y \in C \in \mathcal{C}'$ as k_1 decreases and either (3.2) is not tight for u, v in G, or $f'(x) + f'(y) \geqslant f(x) + f(y) - 1$ (by Claim 1);
* (3.3) holds as k_1 decreases by 1 and the left hand side decreases by at most 2 (by Claim 0).

Once again by minimality of G we get that G' is f'-choosable, and that gives that G is L-colourable, a contradiction. □

We are now ready to derive the final contradiction. If $k_3 = 0$ then G has only parts of size 1 in $\mathcal{S}$ and it is immediate that G is f-choosable. Assume $k_3 \neq 0$. Recall that the total number of colors in all lists is at most $3k_3 + s - 1$. Let $\{u, v, w\}$ be a part of size 3. Then $f(u) + f(v) + f(w) \geqslant 4k_3 + s - 1 > 3k_3 + s - 1$ and therefore there must be a colour c which appears in two out of three colour sets $L(u)$, $L(v)$, $L(w)$, say $c \in L(u) \cap L(v)$.

We colour u and v with c and consider $G' = G - \{u, v\}$ with lists $L'(x) = L(x) - \{c\}$. Again, the partition $\mathcal{S}'$, $\mathcal{C}'$ is inherited from G and we simply put $\mathcal{A}' = (\{w\})$. Thus, $k_1' = 1$, $s' = s$, $k_3' = k_3 - 1$. We verify the inequalities (1)-(3.3) for G' with f'. The inequality (1) for $A_1' = \{w\}$ holds as

$$f'(w) = f(w) > k_3 = k_3' + 1.$$

All the other inequalities hold for analogous reasons as before. Once again, by minimality of G, we get that G' is f'-choosable, and that gives that G is L-colourable, a contradiction. □

The last result of the paper is another immediate consequence of Lemma 5.

Corollary 11. $\mathrm{ch}^{\mathrm{OL}}(K_{3\star k}) \leqslant \frac{3}{2}k$, *for any positive integer* k.

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Theoretical Computer Science Department, Faculty of Mathematics and Computer Science, Jagiellonian University, Kraków, Poland
E-mail address: jkozik@tcs.uj.edu.pl

Theoretical Computer Science Department, Faculty of Mathematics and Computer Science, Jagiellonian University, Kraków, Poland
E-mail address: piotr.micek@tcs.uj.edu.pl

Department of Mathematics, Zhejiang Normal University, China, and Department of Applied Mathematics, National Sun Yat-sen University, Taiwan
E-mail address: xudingzhu@gmail.com