OPUS 24 project proposal
Full research concept

Dimension and Boolean dimension of partial orders

Guidelines. In the following we describe the state of the art, the objectives, and our methodology in each research direction. Several objectives are tagged as *major*: these are long-term goals around which the work will be concentrated. Most of the other objectives are just subgoals, always on the way to a major one. This project is problem-oriented. We aim at turning problems and conjectures into meaningful examples and theorems. The goals are very concrete. This concrete style makes it of course high-risk. Within the proposal we are arguing that the time is ripe to attack these long-standing open problems.

State of the art, objectives, and methodology

Introduction. A *graph* consists of vertices and edges connecting the vertices. A *poset* consists of elements and a binary relation that is reflexive, antisymmetric, and transitive. The concept of modelling relations between objects as in a graph or in a poset is one of the most fundamental in mathematics or theoretical computer science. Its most important applications are probably in statistical physics, discrete optimization, social sciences, and last but not least data and computer science, where we model networks of communication, computational devices, scheduling problems, and many more. Once a theoretical formulation of a real world situation has been established, it is a challenging task to efficiently solve various optimization problems on the resulting instances.

A fundamental process in theoretical computer science is that of partitioning a set of objects into classes according to certain rules. Perhaps the simplest example is the one coming from graph theory: for each pair of vertices we are told whether they are adjacent or not, and the adjacent vertices shall not go to the same class. The minimum number of classes (colors) that we can partition the vertices of a graph G satisfying the rules is the *chromatic number*, denoted by $\chi(G)$. The simplicity of the rules does not mean that the problems encountered are simple—on the contrary. Starting from the four-color problem around 1850 and later the Hadwiger's conjecture, the theory has developed into a many-sided body of problems, results, and applications. Still, the number of simply stated but challenging problems remains large and in fact grows. This explains why the area attracts so many active researchers. During the process the chromatic number became one of the most important measures of complexity of graphs.

The dimension, introduced in 1941 by Dushnik and Miller [15], is a key measure of complexity of posets. The *dimension* $\dim(P)$ of a poset P is the least integer d such that points of P can be embedded into $\mathbb{R}^d$ in such a way that $x \leqslant y$ in P if and only if the point of x is below the point of y with respect to product order of $\mathbb{R}^d$ (the component-wise comparison of coordinates), see Figure 1. Though this definition justifies the geometric intuition behind the notion of dimension, usually we work with its combinatorial equivalent, see Section 2. In fact, the dimension could be seen as the chromatic number of an appropriately defined hypergraph on the set of all incomparable pairs of P. This sets up a parallel that dimension for posets is like the chromatic number for graphs. The decision problem $\chi(G) \leqslant 2$, for graph G on the input, is polynomial, the same as the problem $\dim(P) \leqslant 2$, for poset P on the input. While the decision problems $\chi(G) \leqslant 3$ and $\dim(P) \leqslant 3$ are NP-complete.

How do the graphs with large chromatic number look like? The simplest certificate that a graph needs many colors is a large clique. A *clique* in a graph is a subset of vertices such that every two distinct vertices are adjacent. The *clique number* of a graph G, denoted by $\omega(G)$, is the maximum size of a clique in G. Clearly, $\omega(G) \leqslant \chi(G)$. Yet, the clique number and the chromatic number may be arbitrarily far apart; there are various constructions witnessing this phenomenon: Erdős' random graph [19], Kneser graphs [36], shift graphs (by Erdős and Hajnal [20]), Tutte's construction [10, 11], and perhaps the best-known Mycielski's construction [44]. However, these classical constructions require a lot of freedom in connecting vertices by edges. Many important classes of graphs have chromatic number bounded in terms of clique number. Such classes are called *χ-bounded*. The χ-boundedness is an exciting and very lively field of research. See a recent survey by Scott and Seymour [53]. Chudnovsky, Scott, Seymour and Spirkl recently published [9] a proof of

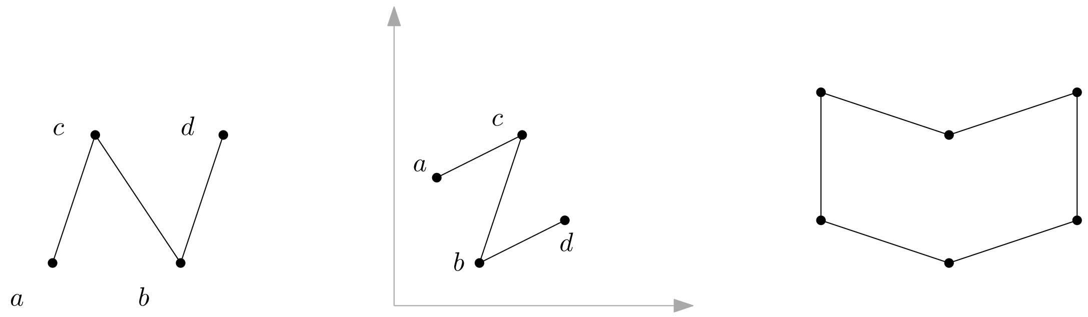

Figure 1: On the left: an example of a 2-dimensional poset and its embedding into $\mathbb{R}^2$. On the right: an example of a 3-dimensional poset.

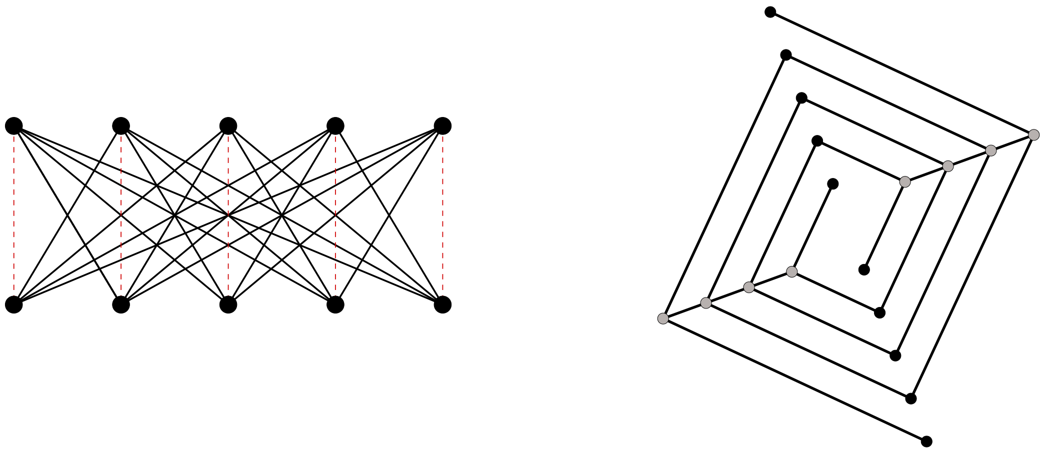

Figure 2: The standard example S_5 (left). Kelly's planar poset containing an induced S_5 (right).

Gyárfás conjecture that graphs with no large induced cycle are χ-bounded. In 2014, we answered [51] an old question of Erdős and showed that intersection graphs of line segments in the plane are not χ-bounded.

How do the posets with large dimension look like? The simplest certificate that a poset has large dimension is a large standard example. For $k \geqslant 2$, the *standard example* S_k is defined as the poset consisting of k minimal elements $a_1, \dots, a_k$ and k maximal elements $b_1, \dots, b_k$, such that $a_i < b_j$ in S_k if and only if $i \neq j$, see Figure 2. Already Dushnik and Miller [15] noted that $\dim(S_k) = k$. The *standard example number* of a poset P, denoted by $\mathrm{se}(P)$, is the maximum size of a standard example in P. Clearly, $\mathrm{se}(P) \leqslant \dim(P)$. Yet, there are classical constructions of posets with no large standard examples and arbitrarily large dimension; for instance interval orders. A poset P is an *interval order* if the elements of P can be represented by intervals of the real line in such a way that $x < y$ in P if and only if the interval of x is completely to the left of the interval of y. It turns out that interval orders are exactly the posets avoiding S_2. Still, the maximum dimension of an interval order of size n grows like $\Theta(\log \log n)$ as shown by Füredi, Hajnal, Rödl, and Trotter [23]. Actually, they discovered a surprisingly tight connection between the chromatic number of shift graphs, the dimension of interval orders and the number of antichains in the lattice of subsets of a finite set. Another classical example is a family of incidence posets. The *incidence poset* I_G of a graph G is the height-2 poset with element set $V(G) \cup E(G)$, where for a vertex v and an edge e we have $v < e$ in I_G whenever v is an endpoint of e in G. Since every maximal element of I_G is above exactly two minimal elements of I_G, the standard example number of any incidence poset is at most 3. On the other hand, the fact that $\dim(I_{K_n}) \geqslant \log \log n$ follows easily from iterated applications of the Erdős-Szekeres theorem for monotone sequences. We call a class of posets $\dim$-*bounded* if there is a function such that for every poset in the class its dimension is bounded by the function of its standard example number. Thus the class of interval orders is not $\dim$-bounded and the class of incidence posets is also not $\dim$-bounded.

Summary of objectives. This project is problem-oriented. We aim to attack and resolve some major challenges in the field. We distinguish four directions of intended research.

► **Structural.** One can say that the most natural structural property of graphs or posets is *planarity*. How do the planar posets with large dimension look like? Is it true that large dimension in a planar poset must be witnessed by a large standard example? What are the unavoidable minors in cover graphs of posets with large dimension?

► **Logic and computation.** Focusing on the encoding aspect, Nešetřil and Pudlák in 1989, proposed a more expressive version of posets dimension, the Boolean dimension. They posed a beautiful open problem: Is the Boolean dimension of planar posets bounded? If true, this would imply a breakthrough result on the theoretical computer science side—an existence of a reachability labelling scheme for planar digraphs with labels of $\mathcal{O}(\log n)$ size. This would be best possible and would improve the labelling scheme given by M. Thorup [JACM 2004]. Finally, we aim to use our structural results to solve algorithmic type of problems. The most important one is to verify if there is a polynomial time algorithm for computing dimension of posets with bounded width.

► **Extremal.** The maximum chromatic number of n-vertex triangle-free graphs is well-understood. What is the natural parallel of that for posets? The most natural poset equivalent is the maximum dimension of n-element posets without standard example of size 3. In the graph setting, the asymptotic behaviour is $\Theta(\sqrt{n/\log(n)})$. For posets, we only know that it is $o(n)$. This upper bound is screaming to be improved.

► **Sparsity.** Sparse classes of graphs are one of the research themes where people with very different backgrounds can meet and exchange ideas. The notions of *bounded expansion* and *nowhere denseness* uncovered deep links between combinatorial, algorithmic, and logical view points. These concepts tie in unexpectedly with combinatorics of posets. The property of a graph class being nowhere dense can be captured by looking at the dimension of posets whose cover graphs are in the class. Can we do the same for classes of bounded expansion?

1 Structural problems

One can say that the most natural structural property of graphs or posets is planarity. Posets are visualized by their *diagrams*: Points are placed in the plane and whenever $a < b$ in the poset, and there is no point c with $a < c < b$, there is a curve from a to b going upwards (that is y-monotone). The diagram represents those relations which are essential in the sense that they are not implied by transitivity, known as *cover relations*. The undirected graph implicitly defined by such a diagram is the *cover graph* of the poset. That graph can be thought of as encoding the 'topology' of the poset.

We study finite posets whose diagram can be drawn in a planar way, called *planar posets*. Unlike planar graphs, planar posets have a rather wild structure. For instance, recognizing planar posets is an NP-complete problem. More importantly for our purposes, planar posets have unbounded dimension, contrasting with the fact that planar graphs are 4-colorable. Kelly [33] in 1981 showed how to embed standard examples into planar posets, see Figure 2. The following questions are long-standing open problems perhaps the most inspiring in whole of poset theory.

► **Major Objective 1.**

1. *Are posets with planar diagrams* $\dim$-*bounded?*
2. *Are posets with planar cover graphs* $\dim$-*bounded?*

We believe the first published reference to Objective 1.1, is an informal comment by William T. Trotter on page 119 of his book [57] published in 1992. However, the question was circulating among researchers soon after the Kelly construction illlustrated in Figure 2 appeared. Accordingly, the first question is more than 40 years old and obviously a positive answer for the second one would be an even stronger statement.

Within the last decade there was a stream of results bounding dimension of posets with sparse cover graphs. One could say though, that it all started back in the 1970's. There was, and remains, a common belief that a poset with a 'nice drawing' or 'not too dense' cover graph should have small dimension. In this vein, Trotter and Moore [58] showed that if the cover graph of a poset P is a forest, then $\dim(P) \leqslant 3$, and this bound is best possible. They also showed that if P is a planar poset with a single minimal element, then $\dim(P) \leqslant 3$. For a brief moment in time in the late 70's, the researchers believed that all planar posets might have dimension at most 4. That could be another '4-color theorem'. This turned out to be completely false, as Kelly [33] showed in 1981 how to embed standard examples into planar posets, see again Figure 2. In view of Kelly's construction, the study of connections between the dimension and the structure of the cover graph was withheld for many years.

A new impulse was given by a breakthrough result of Streib and Trotter [55] that posets with planar cover graphs have dimension bounded by a function of their height. To bring some context of this result, recall that the *width* of a poset P is the largest size of an antichain in P, and the *height* of P is the largest size of a chain in P. In 1950, Dilworth [12] proved that $\dim(P) \leqslant \mathrm{width}(P)$. On the other hand, posets of height 2 may have arbitrarily large dimension, see the standard examples. Rephrasing, for a poset to have large dimension, the poset must be *wide*. A remarkable feature of planar posets, proven by Streib and Trotter, is that if they have large dimension then they are also *tall*. Soon afterwards, it was shown in a sequence of papers that requiring the cover graph to be planar in the Streib-Trotter result could be relaxed: Posets have dimension upper bounded by a function of their height if their cover graphs

- have bounded treewidth, bounded genus, or more generally exclude an apex-graph as a minor [30];
- exclude a fixed graph as a (topological) minor [59, 42];
- belong to a fixed class with bounded expansion [31], see Section 4.

One of the highlights of this series of papers is that every time we came up with a more general result, the arguments were getting conceptually simpler and (often) shorter. The final insight was to connect poset dimension with the weak coloring numbers. Weak coloring numbers were originally introduced by Kierstead and Yang [34] as a generalization of the degeneracy of a graph (also known as the *coloring number*). They are defined as follows. Let G be a graph and consider some linear order π on its vertices (it will be convenient to see π as ordering the vertices of G from left to right). Given a path Q in G, we denote by $\mathrm{left}(Q)$ the leftmost vertex of Q w.r.t. π. Given a vertex v in G and an integer $r \geqslant 0$, we say that $u \in V(G)$ is *weakly r-reachable from v* w.r.t. π if there exists a path Q of length at most r from v to u in G such that $\mathrm{left}(Q) = u$. We let $\mathrm{WReach}_r^\pi[v]$ denote the set of weakly r-reachable vertices from v w.r.t. π (note that this set contains v for all $r \geqslant 0$). The *weak r-coloring number* $\mathrm{wcol}_r(G)$ of G is defined as

$$\mathrm{wcol}_r(G) := \min_\pi \max_{v \in V(G)} |\mathrm{WReach}_r^\pi[v]| \,.$$

The general message, discovered and presented within [29] is that dimension works surprisingly well with weak coloring numbers. The following theorem is the best illustration of this principle.

Theorem. *Let P be a poset of height at most h, let G denote its cover graph, and let $c := \mathrm{wcol}_{3h-3}(G)$. Then*

$$\dim(P) \leqslant 4^c.$$

The proof of this theorem is roughly two-page long. Since we have good bounds on weak coloring numbers of sparse graphs, this result implies all the itemized statements above, and it also gives better bounds than the original results. For example, the $\mathrm{wcol}_h(G)$ for planar G is known to be $\mathcal{O}(h^3)$. Therefore, the theorem above says that height-h posets with planar cover graphs have dimension $2^{\mathcal{O}(h^3)}$, while the initial bounding function given by Streib and Trotter was a long tower.

The best known bounds in the planar case are as follows: the height-h posets with a planar cover graph have dimension $\mathcal{O}(h^6)$, see [37], and the height-h posets with a planar diagram have dimension $\mathcal{O}(h)$, see [32]. These two papers use extensively a very promising new technique that we call *unfolding a poset*. On the level of intuitions it works as follows: if a poset has large dimension, then it has a 'local' subposet which still has large dimension. This type of statement has a very simple and descriptive analogue in the world of graphs: if a graph G is connected and $\chi(G) > 2k$, then at least one distance level L (considered from any fixed vertex v) satisfies $\chi(G[L]) > k$. The 'locality' stems from the fact that in many cases (e.g. in minor-closed classes of graphs) we can handle all the previous distance levels as if they were a single vertex. Extracting such levels in an iterative manner is a powerful tool when one tries to bound the chromatic number of graphs with geometric representations. It was first used by Gyárfás [27] and is known in the community as the Gyárfás trick. Almost all modern proofs bounding the chromatic number have this trick encoded. We believe that we have just developed a poset counterpart of this method.

Objective 1 is the ultimate goal of almost all big projects in dimension theory. Clearly, a large standard example is a much more subtle structure to 'force' than a long chain. Still, we believe that the amount of material (results, proof methods, ad hoc tricks) produced around this topic puts us in the position to claim that the time is ripe for a serious attack.

In particular, together with Heather S. Blake and William T. Trotter we made a significant progress within the last two years and proved our long conjectured intermediate statement for Objective 1.2: the class of posets with a planar cover graph and a unique minimal element is dim-bounded, see [6]. We are very hopeful about expanding this result and reaching the full statement in the future.

The problem whether dimension is tied with the size of the largest standard example is open also for posets whose cover graphs have bounded treewidth or even pathwidth. Together with Gwenaël Joret from Brussels, Michał Pilipczuk from Warszawa and Bartosz Walczak (my local colleague), we are working very hard on this problem. The research tools here seems to be very different than in the planar case. Our proof strategies rely on Colcombet's deterministic version of Simon's factorization theorem, which is a fundamental tool in formal language and automata theory, which we believe deserves wider recognition in structural and algorithmic graph theory.

▶ **Objective 2.** *Dim-boundedness of posets with cover graphs of bounded treewidth / cliquewidth / twinwidth. Prove or disprove.*

As we suggest in the statement, we recognize possible connections even beyong posets with cover of bounded treewidth. The tools we consider seem to be robust enough for bounded cliquewidth case and perhaps even beyond. Connections between dimension theory and the newly developped concept of twinwidth (of graphs) is a possible and very exciting outcome.

Most generally, the dim-boundedness question could be positive for posets with cover graphs excluding a fixed graph as a minor. The latter would imply a positive answer for all previous statements. On the other hand, the property does not hold for posets whose cover graphs have bounded degree (so exclude a fixed graph as a topological minor).

2 Logic and computation

We start with a combinatorial definition of poset dimension. A *linear extension* L of a poset P is a total order on the elements of P such that if $x \leqslant y$ in P then $x \leqslant y$ in L. A *realizer* of a poset P is a set $\{L_1, \ldots, L_d\}$ of linear extensions of P such that

$$x \leqslant y \text{ in } P \iff (x \leqslant y \text{ in } L_1) \wedge \cdots \wedge (x \leqslant y \text{ in } L_d),$$

for every $x, y \in P$. Then, the *dimension* of P, denoted by $\dim(P)$, is the minimum size of its realizer.

The realizers provide a way of succinctly encoding posets. Indeed, if a poset is given with a realizer witnessing dimension d, then a query of the form "is $x \leqslant y$?" can be answered by looking at the relative position of x and y in each of the d linear extensions of the realizer. This application motivates the following presumably more efficient encoding of posets proposed by Nešetřil and Pudlák [49], following the work of Gambosi, Nešetřil and Talamo [24].

A *Boolean realizer* of a poset P is a set of linear orders (not necessarily linear extensions) $\{L_1, \ldots, L_d\}$ of elements of P for which there exists a d-ary Boolean formula ϕ such that

$$x \leqslant y \text{ in } P \iff \phi((x \leqslant y \text{ in } L_1), \ldots, (x \leqslant y \text{ in } L_d)) = 1,$$

for every $x, y \in P$. The *Boolean dimension* of P, denoted by $\operatorname{bdim}(P)$, is a minimum size of a Boolean realizer. Clearly, for every poset P we have $\operatorname{bdim}(P) \leqslant \dim(P)$.

The usual dimension of a poset on n elements may be linear in n as witnessed by standard examples: $\dim(S_n) = n$. It is a nice little exercise to show that $\operatorname{bdim}(S_n) \leqslant 4$ for every n. In general, Nešetřil and Pudlák [49] showed that Boolean dimension of posets on n elements is $O(\log n)$. They also provide an easy counting argument showing that there are posets on n elements with Boolean dimension at least $c \log n$, where c is some positive constant.

As we have already seen, planar posets may have arbitrarily large dimension. Nešetřil and Pudlák proposed a beautiful problem that remains a challenge with essentially no progress over 30 years. This shall be one of our main goals.

▶ **Major Objective 3.** *Resolve the problem: Is the Boolean dimension of planar posets bounded?*

Nešetřil and Pudlák suggested an approach for a negative resolution of this question that involves an auxiliary Ramsey-type problem for planar posets. However, no progress in this direction has been made. From the positive side, researchers have in recent years investigated conditions on cover graphs that bound Boolean dimension. In 2017, together with S. Felsner and T. Mészáros [22] we finished a long effort and proved that posets with cover graphs of bounded treewidth have bounded Boolean dimension. On the other hand, as noted previously, the Kelly examples have unbounded dimension, but their cover graphs have pathwidth at most 3. Finally, the latest breakthrough, proved together with H. Blake and W. Trotter [6] is as follows: if P is a poset with a planar cover graph and a unique minimal element, then

$$\operatorname{bdim}(P) \leqslant 13.$$

Boolean realizers have a natural connection to labeling schemes for reachability queries.

Given two vertices u, v in a directed graph (digraph, for short), we say that u can *reach* v if there is a directed path from u to v in the digraph. A class of digraphs $\mathcal{C}$ admits an $f(n)$*-bit reachability scheme* if there exists a function $A : (\{0,1\}^*)^2 \to \{0,1\}$ such that for every positive integer n and every n-vertex digraph $G \in \mathcal{C}$ there exists $\ell : V(G) \to \{0,1\}^*$ such that $|\ell(v)| \leqslant f(n)$ for each vertex v of G, and such that, for every two vertices u, v of G

$$A(\ell(u), \ell(v)) = \begin{cases} 1 & \text{if } u \text{ can reach } v \text{ in } G; \\ 0 & \text{otherwise.} \end{cases}$$

Bonamy, Esperet, Groenland, and Scott [7] have recently devised a reachability labeling scheme for the class of *all* digraphs using labels of length at most $n/4 + o(n)$. This is best possible up to the lower order term as a simple counting argument forces every reachability labeling scheme for the class of *all* n-vertex digraphs to use a label of length at least $n/4 - o(n)$.

In 2004, M. Thorup [56] presented an $\mathcal{O}(\log^2 n)$-bit reachability labeling scheme for planar digraphs. It remains open to answer whether more efficient schemes exist.

► **Major Objective 4.** *Verify if planar digraphs admit an $\mathcal{O}(\log n)$-bit reachability labeling scheme. If possible, improve the labeling scheme given by Thorup in 2004.*

There is a standard technique for reducing reachability queries in digraphs to comparability queries in posets: Given a digraph G, contract each strongly connected component of G to a single vertex. Let G' be the resulting digraph which is obviously acyclic. Note also that if G is planar, then G' is planar as well. Now given a labeling of G' we extend it to a labeling of G by assigning to each vertex v of G the label of the strong component of v in the labeling of G'. Within an acyclic digraph G', let $u \leqslant v$ if u can reach v in G' for all u, v in G'. Clearly $(G', \leqslant)$ forms a partially ordered set. Thus, if we have an $f(n)$-bit comparability labeling scheme for posets with planar cover graphs, then we immediately get an $f(n)$-bit reachability scheme for general planar digraphs.

Note that if $\mathcal{C}$ is a class of posets with bounded Boolean dimension, then $\mathcal{C}$ admits an $\mathcal{O}(\log n)$ comparability labeling scheme. To see this, suppose that $\operatorname{bdim}(P) \leq d$ for every poset P in $\mathcal{C}$. Now let P be in $\mathcal{C}$, and let $(L_1, \ldots, L_d)$ with a formula ϕ be a Boolean realizer of P. Let n be the number of elements of P. We label each element $x \in P$ with a bitstring of length $d \cdot \lceil \log n \rceil$ describing the positions of x in $(L_1, \ldots, L_d)$. Now given labels for two elements $x, y \in P$ and the formula ϕ, we can determine if $x \leqslant y$ in P. The formula ϕ is a function from $(\{0,1\}^d)^2$ to $\{0,1\}$, so there are only $2^{2^{2d}}$ possibilities, and these can be encoded with an additional 2^{2d} bits in each label.

Therefore, a positive resolution of Objective 3 would give us the desired $\mathcal{O}(\log n)$ labelling in Objective 4 but perhaps there is also a way to attack Objective 4 directly, just using our toolbox.

Finally, we would like to apply our structural results and proof techniques to attack algorithmic type of questions. In spite of its central importance in the theory of posets, understanding of the computational aspects of dimension has lagged behind that of its combinatorial properties and of its relation to other parameters of posets. The computational complexity of determining the dimension of a poset was one of the twelve outstanding open problems in Garey and Johnson's treatise on NP-completeness [25]. It has been proved NP-complete independently by Lawler and Vornberger [39] and by Yannakakis [60]. Actually, Yannakakis proved the stronger statement that deciding whether the dimension is at most d is NP-complete for every fixed $d \geqslant 3$, using a reduction from graph 3-colorability. On the other hand, the results of Dushnik and Miller's pioneering work [15] easily imply that one can test in polynomial time whether the dimension is at most 2 (McConnell and Spinrad [41] gave a linear-time algorithm). Spinrad [54] gives an account of several other computational aspects of poset dimension theory.

Approximating the dimension is also hard. Chalermsook, Laekhanukit, and Nanongkai [8] showed that unless NP $=$ ZPP no polynomial time algorithm exists that approximates the dimension of a poset within a factor of $O(n^{1-\epsilon})$ for any $\epsilon > 0$, which improves an earlier result of Hegde and Jain [28] on hardness of $O(n^{1/2-\epsilon})$-approximation.

How about parameterized algorithms for dimension? The two natural choices for parameters are the height and the width of the poset. For the height, Yannakakis [60] showed that deciding whether a poset of height 2 has dimension at most d is NP-complete for every fixed $d \geqslant 4$. The decision problem for posets of height 2 whether they have dimension at most 3 was open over 30 years and was recently proved to be NP-complete as well, by Felsner, Mustaţă and Pergel [21].

On the other hand, the question about complexity of computing dimension for posets of bounded width seems to be a true mystery. Möhring [43] proposed this question already in the 1980s. Since for every poset P we have $\dim(P) \leqslant \mathrm{width}(P)$, and deciding whether $\dim(P) \leqslant 2$ is polynomial, the question starts to be interesting for posets P with $\mathrm{width}(P) \geqslant 4$. The NP-hardness proof by Yannakakis does not cover this case, as it uses partial orders whose width grows with the size of the instance.

► **Objective 5.** *Understand the algorithmic implications of bounded width for the computing dimension problem.*

The straightforward resolution would be to verify whether there is a polynomial-time algorithm to determine the dimension of a poset of bounded width.

We might approach this objective from the point of view of parameterized complexity theory. A problem with parameter k is called *fixed-parameter tractable* if it can be decided in time bounded by $f(k)n^{\alpha}$ for some (computable) function f of the parameter and some absolute constant α. The following is somewhat a weaker algorithmic task to verify: Is there an algorithm computing the dimension of a poset of size n and width w in time $f(w)n^{\alpha}$ for some function f and constant α? Especially, a positive answer to one of these questions most likely requires deep structural properties of large dimensional posets with bounded width. Another possible relaxation of last objectives is to give up exact computation of dimension and attempt for good approximation algorithms. Since the general problem of computing dimension is very hard to approximate, the assumption of bounded width would again be the key to allow some positive results.

One way to approach these algorithmic problems is to restrict the input to specific but important classes of posets: interval orders, $(k+k)$-free posets, and posets with bounded interval dimension (i.e. multidimensional analogues of interval orders). The incomparability graphs of interval orders of bounded width have bounded pathwidth. Therefore, an approach to the problem for interval orders, which we would like to try first, is to use dynamic programming on the incomparability graph. There are no doubts that the algorithmic side of the project will be successfull only if we manage to build good structural results in the background.

3 Extremal problems

How can global parameters of graphs or posets such as edge densities or chromatic number/dimension, influence its local substructures? For instance, how large is the maximum chromatic number of n-vertex triangle-free graphs? Note that in this setup, we do not impose any global structural properties (like the planarity or any kind of sparsity). Questions of this type are among the most natural ones and historically well-studied; perhaps starting from Turán's Theorem. Collectively, these problems are known as extremal graph theory.

As already mentioned, there are classical constructions of graphs with unbounded chromatic number and no large clique. The maximum chromatic number of n-vertex triangle-free graphs is well-known to be $\Theta(\sqrt{n/\log n})$ and it follows from the results on the off-diagonal Ramsey numbers $R(3,t)$ given by Ajtai, Komlós, and Szemerédi [2] and Kim [35].

What is the maximum dimension of n-element posets without S_2 (standard example of size 2)? This is already a non-trivial question. While the case of graphs with no cliques of size 2 is trivial. The posets excluding S_2 are known to be *interval orders* and as mentioned the maximum dimension of interval orders of size n grows like $\Theta(\log\log n)$ as shown by Füredi et al. [23]. Let $f(d,n)$ be the maximum dimension of posets on at most n elements with no S_d as a subposet. Combining results of several authors, see the discussion in [5], the value of $f(2,n)$ can be determined to within an additive error of at most 5. For many years there was no non-trivial upper bound for $f(d,n)$ when $d \geqslant 3$. Trivially, when P has at most n elements then $\dim(P) \leqslant n$. Only in 2015, Biró, Hamburger, and Pór [4] proved that $f(d,n) = o(n)$, for every fixed $d \geqslant 3$. In fact, the upper bound is of the form $n/\operatorname{polylog} n$. From the lower bound side there is a brand new manuscript by Biró et al. [3] who prove that for every $d \geqslant 2$ there exists $\alpha_d > 0$ such that $f(d,n) = \Omega(n^{\alpha_d})$. Their work and proof techniques are based on a thorough study of random bipartite posets given by Erdős, Kierstead, and Trotter [18]. We believe that the same kind of polynomial bound should hold from above.

► **Major Objective 6.** *For every $d \geqslant 3$ verify whether n-element posets without S_d have dimension $\mathcal{O}(n^{\alpha})$ for some $\alpha < 1$.*

The case of $d = 3$ is particularly interesting. We believe that it represents the whole difficulty of the problem.

► **Objective 7.** *Improve the lower and upper bounds for the maximum dimension of n-element posets without S_3. The current state of art is: $\Omega(n^{1/6})$ and $o(n)$.*

These objectives are related to questions on stability analysis, i.e. one of the flavours of the extremal side of graph theory. There is a lot of progress on this topic for posets, see [3].

Given an integer $c > 0$ and an n-vertex graph G with $\chi(G) \geqslant n - c$, what can we say about the clique number of G? It is a rather easy exercise to show that G must have a clique of size at least $n - 2c$. Indeed, while G is not a complete graph, choose two non-adjacent vertices and remove them. Each such operation decreases the size of the graph by 2 but lowers the chromatic number by at most 1. So the operation must halt in at most c steps. Turning back to posets. Given an integer $c > 0$ and a $(2n + 1)$-element poset P with $\dim(G) \geqslant n - c$, what can we say about the standard example number of P? Let $g(c)$ be the minimum value such that $\mathrm{se}(P) \geqslant n - g(c)$. The best known asymptotic bounds for $g(c)$ are $\Omega(c^{3/2}/\operatorname{polylog}(c))$ and $\mathcal{O}(c^2)$, and they come from [5, 3], respectively. We believe that the same insights are needed to improve the bounds for $g(c)$ and to make progress on Objective 6.

► **Objective 8.** *Improve the bounds for the stability analysis of posets dimension.*

4 Sparsity

Structural graph theory has expanded far beyond the study of classes of graphs that exclude a fixed minor. One of the driving forces was, and is, to develop efficient algorithms for computationally hard problems for graphs that are 'structurally sparse'. The key concepts in the area were introduced by Nešetřil and Ossona de Mendez in [46] and [48].

A class of graphs is *nowhere dense* if for every $r \geqslant 1$, there exists $t \geqslant 1$ such that no graph in the class contains a subdivision of the complete graph K_t where each edge is subdivided at most r times as a subgraph. A class of graphs has *bounded expansion* if for every $r \geqslant 1$, there exists $c \geqslant 0$ such that no graph in the class contains a subdivision of a graph with average degree at least c where each edge is subdivided at most r times as a subgraph.

These are very robust properties that include every class excluding a fixed graph as a minor, but also classes of graphs that allow drawings with bounded number of crossings per edge, or graphs of bounded book-thickness, see [50]. At first sight, bounded expansion might seem to be a weak property for a graph class. Yet, this notion captures enough structure to allow solving a wide range of algorithmic problems efficiently: Dvořák, Král' and Thomas [16] devised an FPT algorithm for testing first-order definable properties in classes of bounded expansion. This was later extended to an almost linear algorithm for testing FO properties in nowhere dense classes by Grohe, Kreutzer and Siebertz [26].

One reason these particular definitions attracted so much attention in recent years is the realization that they can be verbalized in several, seemingly very different ways. Algorithmic applications in turn typically build on the 'right' characterization for the problem at hand and sometimes rely on multiple ones. Nowhere dense classes were characterized in terms of shallow minor densities [48] and consequently in terms of generalized coloring numbers (by results from [61]), low tree-depth colorings [48] (by results from [46]); they were also characterized in terms of quasi-uniform wideness [47, 38, 52], the so-called splitter game [26], sparse neighborhood covers [26], neighborhood complexity [17], the model theoretical notion of stability [1], as well as existence of particular analytic limit objects [45]. Finally, there is a characterization in terms of poset dimension [29] that established a strong bond between the modern research in both fields. For a class of graphs $\mathcal{C}$ closed under taking subgraphs we proved that $\mathcal{C}$ is nowhere dense if and only if for every $h \geqslant 1$ and real number $\varepsilon > 0$, n-element posets of height at most h whose cover graphs are in class have dimension $\mathcal{O}(n^{\varepsilon})$.

Most of the characterizations of nowhere dense classes of graphs go hand in hand with respective characterizations of classes with bounded expansion. For example, Zhu [61] proved that: (1) A class has bounded expansion if and only if for every $r \geqslant 0$, there exists $c \geqslant 1$ such that every graph in the class has weak r-coloring number at most c; (2) A class is nowhere dense if and only if for every $r \geqslant 0$ and every $\epsilon > 0$, every n-vertex graph in the class has weak r-coloring number $\mathcal{O}(n^{\epsilon})$. Often, it is the case that when a new characterization is discovered, it is first done for classes with bounded expansion, and a respective statement is conjectured to be true for nowhere dense classes. Then, after a year or two, the statement for nowhere dense is proved to be true, likely with a very different (more insightful) proof.

In the case of the characterization of nowhere dense classes in terms of poset dimension, first we had one of the implcations for the bounded expansion case, see the 4^c-bound stated in Section 1. That bound together with Zhu's characterization gives that posets with cover graphs in a fixed class of bounded expansion have dimension bounded in terms of their height. The 4^c-bound falls short of implying the analogous result for nowhere dense classes. Indeed, if the cover graph G has n vertices and belongs to a nowhere dense class, we only know that $\mathrm{wcol}_{3h-3}(G) \in \mathcal{O}(n^{\epsilon})$ for every $\epsilon > 0$. Thus, we only deduce that $\dim(P) \leqslant 4^{\mathcal{O}(n^{\epsilon})}$ for every $\epsilon > 0$, which is a vacuous statement since $\dim(P) \leqslant n$ always holds. The result implying $\mathcal{O}(n^{\varepsilon})$-bound for posets with cover graphs in a fixed nowhere dense class required new ideas. The key new insight is Lemma 7 in [29] that allows to find a somehow local piece of the poset responsible for large dimension. This lemma proved to be a useful tool; it is an important step in the first polynomial bound for poset dimension in the planar cover graph case, see [37].

The backward implication for the characterisation of nowhere dense classes appeared to be an easy and routine argument. The natural counterpart for classes with bounded expansion is still open.

► **Major Objective 9.** *Characterize classes of graphs with bounded expansion in terms of poset dimension.*

The conjecture is that a monotone class of graphs $\mathcal{C}$ has bounded expansion if and only if for every $h \geqslant 1$, posets of height at most h whose cover graphs are in $\mathcal{C}$ have bounded expansion. The missing implication is the one which in perhaps all known characterisations is the easy or even trivial one. We believe that behind this phenomenon, there is a tough underlying question in poset theory. The critical statement to verify is the following.

► **Objective 10.** *Let $\mathcal{C}$ be a monotone class of bipartite graphs. If, seeing the graphs in $\mathcal{C}$ as posets of height at most 2, these posets have bounded dimension, then the graphs in $\mathcal{C}$ have bounded average degree.*

This last problem is related to a conjecture by T. Łuczak [40]: Let P be a poset of height 2, with dimension at least $d \geqslant 13$ and girth at least d^d. Then for every $P' \subseteq P$, we have $\dim(P') \leqslant \dim(P)$.

Concept and work plan

Among the 10 objectives listed in the description, there are problems that over the years proved to be challenging. Some of them are critical for the whole field. Below we provide our insights and source of hopes that we can make a progress in each direction.

Concerning Objective 1, so dim-boundedness problems in planar case, we should note first that there is no consensus in the community whether the answer should be 'yes' or 'no'. However, within last two years we developed strong intuitions towards a positive resolution. This problem is somehow our fixed point in all our research sessions with William T. Trotter. During his last stay in Kraków (6 months in 2022) we spent countless hours discussing possible angles of attack. For now, we believe that our recent solution for posets with planar cover graphs and a unique minimal element [6] contains ideas that are robust enough to be important in the big picture. Namely, aforementioned unfolding of a poset which we see as an analogue of BFS of a graph and also combinatorics of shadows and shadow blocks that we introduced in [6]. We are determined to intesify this research and attack the full problem in the near future. Hopefully with a support of some youngsters hired thanks to this project proposal.

William T. Trotter is one of the founding fathers of the modern poset theory. He was also a collaborator of Paul Erdős—one of the most prolific mathematicians and producers of mathematical conjectures of the 20th century. The PI of this proposal is proud to collaborate with Trotter (10 joint papers) and their work record so far allows to be optimistic about further breakthroughs. Trotter is mostly interested in Objectives 1 and 3.

Concerning Objective 2, so dim-boundedness of posets with cover graphs of bounded treewidth and beyond, we have significant preliminary research in place. Together with Gwenaël Joret from Brussels and Bartosz Walczak from Kraków, already a couple years ago, we worked out the proof for posets with bounded pathwidth. Since then we were trying to lift the argument from path decompositions to tree decompositions but it proved to be a challenge. Recently, we shared our insights with Michał Pilipczuk from Warszawa who suggested that our tools resemble a particular powerful blackbox from automata theory, namely the Colcombet's deterministic version of Simon's factorization theorem. It is a fundamental tool in formal language and automata theory, which we believe deserves wider recognition in structural and algorithmic graph theory. We are very hopeful about future progress in this direction and we keep in touch with respect to this research. Michał is going to visit us again in the first week of January 2023.

A positive resolution of Objective 4 would be probably the biggest impact of this project. Mikkel Thorup designed a reachability labelling scheme for digraphs with label lengths in $\mathcal{O}(\log^2 n)$ in 2004 and since then the question of a scheme with $\mathcal{O}(\log n)$ remains a challenge. But not for the lack of trying. Any improvement of $\mathcal{O}(\log^2 n)$ bound on label lengths in this case would be very likely a significant result broadly recognized in the theoretical science community. We believe that our angle of attack of Objective 4 through the positive resolution of Objective 3 is original and was not tested enough before. There is also one additional reseource, perhaps enabling a direct approach to Objective 4, the product structure theorem for planar graphs. This statement that we proved in 2019, see [14], turned out to be immensely applicable and paved way for solutions of a number of long-standing open problems, including: a constant upper bound on the queue-number of planar graphs, low polynomial bound for p-centered chromatic numbers of planar graphs, and also most relevant for this proposal an assymptotically optimal adjacency labelling scheme for planar graphs, see [13]. Perhaps one can also push from this direction towards reachability labelling schemes.

Concerning the problem on computational complexity of computing dimension of bounded width posets, i.e. Objective 5, our idea to attack this problem is to consider it in the context of the fixed-parameter tractability when parameterized by the width. We would like to approach it first by restricting the input to specific but important classes of posets: interval orders, $(k + k)$-free posets, and posets with bounded interval dimension (i.e. multidimensional analogues of interval orders). The incomparability graphs of interval orders of bounded width have bounded pathwidth. Therefore, an approach to the problem for interval orders, which we would like to try first, is to use dynamic programming on the incomparability graph, which if successful would result in an FPT algorithm running in time $f(w)n$, where w is the width of the input poset. Another possible relaxation of Objective 5 is to give up exact computation of dimension and attempt for good approximation algorithms. Since the general problem of computing dimension is very hard to approximate, this could yield interesting results in the context of posets of bounded width.

Objectives 6 and 7 are fresh and the PI did not work on them before. The problem though feels fundamental and it is starking how little is known (only the $o(n)$ upper bound!). The initial approach is definitely to avoid the Ramsey-type argument present in the upper bound argument in [4].

Finally, the best way to attack Objective 9 is to develop good understanding around Objective 10. This is a line of work we developed with Gwenaël Joret from Brussels, Stefan Felsner from Berlin and his team. So far, the critical tool were iterated applications of the unfolding (so analogues of BFS crafted for posets). However, succesful attack on this problem definitely requires new insights. Perhaps the problem proposed by T. Łuczak on height-2 posets of large girth could initiate some independent ideas.

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